

The (local) lifting problem for curves

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Abstract.

The *lifting problem* that we consider asks: given a smooth curve in characteristic p and a group of automorphisms, can we lift the curve, along with the automorphisms, to characteristic zero? One can reduce this to a local question (the so-called *local lifting problem*) involving continuous group actions on formal power series rings. In this expository article, we overview much of the progress that has been made toward determining when the local lifting problem has a solution, and we give a taste of the work currently being undertaken. Of particular interest is the case when the group of automorphisms is cyclic. In this case the lifting problem is expected to be solvable—this is the *Oort conjecture*.

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§1. Introduction

Throughout this paper, p represents a prime number, k is an algebraically closed field of characteristic p , and $W(k)$ is the ring of Witt vectors of k , that is, the unique complete discrete valuation ring (DVR) in characteristic zero with uniformizer p whose residue field is k . If $R/W(k)$ is a finite extension, then it will be assumed that R is a DVR (i.e., R is integral). We note that any such R has residue field k .

1.1. What is the lifting problem?

Many problems in mathematics involve starting with some kind of data in characteristic p , and trying to “lift” the data to characteristic zero (i.e., trying to find some analogous data in characteristic zero that reduce to the given data in characteristic p). This paper considers the problem where the data in question are a curve and a finite group of automorphisms. More precisely, we have the following:

Question 1.1. [Lifting problem] Let Y be a smooth, proper curve over k . Let G be a finite group acting faithfully on Y by k -automorphisms. Does there exist $R/W(k)$ finite, and a flat relative curve $Y_R \rightarrow \operatorname{Spec} R$ such that

- (1) $Y_R \times_R k \cong Y$,
- (2) There is an action of G on Y_R by R -automorphisms such that the restriction of this action to the special fiber is the original G -action on Y ?

If the answers to (1) and (2) are “yes,” then we say that the G -action on Y (or the curve Y with G -action) *lifts to characteristic zero* (or *lifts over R*), and that Y_R (with G -action) is a *lift* of Y (with G -action).

One might ask: does every G -action on every k -curve lift to characteristic zero? In fact, the answer is “no.” It is well known (see, e.g., [Har77, IV, Ex. 2.5]) that the number of automorphisms of a curve of genus $g \geq 2$ in characteristic zero is at most $84(g-1)$. But in characteristic p , a curve can have more automorphisms. Roquette [Roq70] gave the example of the smooth, projective model of the curve Y given by $y^2 = x^p - x$. This curve has genus $(p-1)/2$ and $2p(p^2-1)$ automorphisms, which exceeds $84(\frac{p-1}{2}-1)$ for $p \geq 5$. So clearly the canonical $\operatorname{Aut}(Y)$ action on Y cannot lift to characteristic zero.

For another example, it is well known that the automorphism group of \mathbb{P}_L^1 is $PGL_2(L)$ for any field L . If $Y \cong \mathbb{P}_k^1$, then the group $G = (\mathbb{Z}/p)^n$ (for any n) is a subgroup of $PGL_2(k)$, and thus acts faithfully on Y . If this G -action on Y lifts to characteristic zero, then G must act on \mathbb{P}_K^1 for some characteristic zero field K , that is, $G \subseteq PGL_2(K)$. But if $n > 1$ and $p^n \neq 4$, then $PGL_2(K)$ does not contain $(\mathbb{Z}/p)^n$, so the G -action on Y cannot lift to characteristic zero.

This of course leads one to ask: *Can we find necessary and sufficient criteria for a G -action on Y to lift to characteristic zero? Over some particular R ?*

A group G for which any G -action on any curve lifts to characteristic zero is called an *Oort group* for k ([CGH08]).

1.2. What is the local lifting problem?

Maintain the notation of Question 1.1. It turns out that there is a local-global principle for lifting (Theorem 3.1). To wit, if we can lift the germs of the curve Y at each of the points where G acts with inertia to characteristic zero, along with the action of the inertia groups, then we can lift the G -action on Y to characteristic zero. Clearly, the converse holds as well. So if we understand this *local lifting problem*, we can understand the lifting problem. More specifically, if $I_y \subseteq G$ is an inertia group at some $y \in Y$, then I_y acts on the complete local ring of y in Y , which is isomorphic to $k[[u]]$, by continuous k -automorphisms. If there is a lift Y_R of Y with G -action to characteristic zero, then I_y also acts on the complete local ring of y in Y_R , which is isomorphic to $R[[U]]$.

In our description of the local lifting problem below, the letter G can be thought of as one of these inertia groups I_y . The action of G on $k[[u]]$ is called a *local G -action*.

Question 1.2. [Local lifting problem] Let G be a finite group, and suppose we have an embedding $\iota : G \hookrightarrow \text{Aut}_{k, \text{cont}}(k[[u]])$. Does there exist $R/W(k)$ finite, and an embedding $\iota_R : G \hookrightarrow \text{Aut}_{R, \text{cont}}(R[[U]])$, such that, if u is the reduction of U , then the action of G on $R[[U]]$ reduces to that of G on $k[[u]]$?

If the answer is “yes,” then we say that the local G -action *lifts to characteristic zero* (or *lifts over R*), and the G -action on $R[[U]]$ is a *lift* of the G -action on $k[[u]]$. For a reformulation in terms of Galois theory, see Question 4.1.

Since the lifting problem does not always have a solution, neither does the local lifting problem. So we ask: *Can we find necessary and sufficient criteria for a local G -action to lift to characteristic zero? Over some particular R ?*

A group G for which every local G -action on $k[[u]]$ lifts to characteristic zero (and for which there exists a faithful local G -action) is called a *local Oort group* for k .

1.3. Outline

In §2, we give some basic results on the lifting problem and relate it to algebraic fundamental groups of curves. In §3 we reduce the lifting problem to the local lifting problem, which we focus on for the rest of the paper. In §4, we rephrase the local lifting problem in terms of Galois theory, and state some basic results on Galois extensions of $k[[t]]$. Of particular importance is Lemma 4.3, which gives the differentials of some such extensions in terms of their higher ramification filtrations. We also state a weaker form of the local lifting problem, the *birational* local lifting problem, and sketch a proof that it can always be solved, unlike the (standard) local lifting problem. Lastly, we discuss the relationship between solutions to the local lifting problem and solutions to its birational variant.

The heart of the paper begins with §5, where we discuss obstructions to the existence of solutions to the local lifting problem. The most important of these obstructions is the *KGB obstruction* ([CGH09]), for which we compute some examples in great detail. We also give some examples where the KGB obstruction vanishes and where local lifting either is possible or is thought to be possible. The remainder of the paper focuses on two interesting such examples. The case of lifting a *cyclic* local G -action from characteristic p to characteristic zero is the subject of §6. Such local actions are always expected to lift (this is the *Oort conjecture*, see Conjecture 5.4), and have been shown to lift when $p^3 \nmid |G|$ ([OSS89] and [GM98]). We give an overview of the existing work in §6.1–6.5, and in §6.6, we outline a new approach of Wewers and the author to the conjecture. A success of this new approach has been to show that cyclic local G -actions lift to characteristic zero when $p^4 \nmid |G|$ ([OW11]). In §7, we look at the case where $G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$ is a nonabelian group. This case is well-understood due to the work of Bouw, Wewers, and Zapponi, and we give a summary of their work. We also remark on the obstacles to generalizing to the case where G has a larger p -Sylow subgroup.

Appendix A gives background on the non-archimedean geometry used in the paper, but is by no means a comprehensive reference. The reader who has a basic familiarity with formal/rigid geometry should only need to refer to it for notational purposes, whereas the reader without such familiarity will need to read it to have a reasonable understanding of much of §3, §6, and §7. Appendix B discusses the concepts of

depth and *deformation data*, which may be unfamiliar to most readers. However, looking at this appendix may be safely postponed until the paper explicitly references it.

We note that the only other major exposition of the lifting and local lifting problems of which we are aware is Brewis's master's thesis ([Bre05]). The aspects on which [Bre05] focuses are fairly complementary to those on which we focus here. In particular, [Bre05] gives many details on global techniques, the local-global principle, and the birational local lifting problem, whereas we spend the majority of our time discussing specific examples of the local lifting problem.

1.4. Notation and conventions

If Γ is a group of automorphisms of a ring A , we write A^Γ for the fixed ring under Γ . For a finite group G , a *G-Galois extension* (or *G-extension*) of rings is a finite extension $A \hookrightarrow B$ (also written B/A) of integrally closed integral domains such that the associated extension of fraction fields is G -Galois. We do *not* require B/A to be étale.

If x is a scheme-theoretic point of a scheme X , then $\mathcal{O}_{X,x}$ is the local ring of x in X . If R is any local ring, then \hat{R} is the completion of R with respect to its maximal ideal. A *branched cover* $f : Y \rightarrow X$ is a finite, generically étale morphism of geometrically connected, normal schemes. A *G-Galois cover* (or *G-cover*) is a branched cover with an isomorphism $G \cong \text{Aut}(Y/X)$ such that G acts transitively on each geometric fiber of f . Note that G -covers of affine schemes give rise to G -extensions of rings, and vice versa.

Suppose $f : Y \rightarrow X$ is a branched cover, with X and Y locally noetherian. If $x \in X$ and $y \in Y$ are smooth codimension 1 points such that $f(y) = x$, then the *ramification index* of y is the ramification index of the extension of complete local rings $\hat{\mathcal{O}}_{X,x} \rightarrow \hat{\mathcal{O}}_{Y,y}$. If f is Galois, then the *branching index* of a smooth codimension 1 point $x \in X$ is the ramification index of any point y in the fiber of f over x . If $x \in X$ (resp. $y \in Y$) has branching index (resp. ramification index) greater than 1, then it is called a *branch point* (resp. *ramification point*).

If R is any ring with a non-archimedean absolute value $|\cdot|$, then $R\{T\}$ is the ring of power series $\sum_{i=0}^{\infty} c_i T^i$ such that $\lim_{i \rightarrow \infty} |c_i| = 0$. If R has characteristic zero and residue characteristic p (that is, if $0 < |p| < 1$), then we normalize the valuation on R and on $K = \text{Frac}(R)$ so that p has valuation 1.

If X is a smooth curve over a complete discrete valuation field K with valuation ring R , then a *semistable* model for X is a relative curve $X_R \rightarrow \text{Spec } R$ with $X_R \times_R K \cong X$ and semistable special fiber (i.e., the

special fiber is reduced with only ordinary double points for singularities).

Suppose S is a ring of characteristic zero, with an ideal I such that S/I has characteristic p . If an indeterminate in S is given by a capital letter, our convention (which we will no longer state explicitly) will be to write its reduction in S/I using the respective lowercase letter. For example, if $I \subseteq W(k)[[U]]$ is the ideal generated by p , then $W(k)[[U]]/I \cong k[[u]]$, and u is the reduction of U .

If B/A is a finite extension of Dedekind rings, the *degree of the different* of B/A is its length as a B -module. If L/K is a finite extension of discrete valuation fields with valuation rings $A \subseteq K$ and $B \subseteq L$, then the degree of the different of L/K is the degree of the different of B/A .

The group D_n is the dihedral group of order $2n$.

§2. Preliminary global results

Before asking about lifting curves with automorphisms with characteristic zero, it makes sense to ensure that the curves themselves lift to characteristic zero, without worrying about automorphisms. Luckily, this is true:

Proposition 2.1. *The trivial group is an Oort group. Furthermore, the lifting of any curve with trivial action can be done over $W(k)$.*

Proof. Since Y is a curve, the cohomology $H^2(Y, \mathcal{T}_Y)$ is trivial, where \mathcal{T}_Y is the tangent sheaf of Y . By [SGA1, III, Corollaire 6.10], there is a smooth formal relative curve $\mathcal{Y}/W(k)$, whose special fiber is Y . Since, in addition, $H^2(Y, \mathcal{O}_Y)$ is trivial, [SGA1, III, Prop. 7.2] shows that \mathcal{Y} is in fact the formal completion of an *algebraic* relative curve $Y_{W(k)}/W(k)$ at the special fiber. Then $Y_{W(k)}$ is the lift we seek. \square

As the next proposition shows, Proposition 2.1 allows us to shift our perspective from lifting *curves* to lifting *branched covers*. Let G act faithfully on Y , let $X \cong Y/G$, and let $f : Y \rightarrow X$ be the canonical map. Suppose $R/W(k)$ is finite. Then lifting the G -action on Y over R is the same as lifting the G -cover f to a G -cover $f_R : Y_R \rightarrow X_R$, where Y_R and X_R are lifts of the curves Y and X over R (see §1.4 for the definition of G -cover).

Proposition 2.2. (i) *If G acts on Y with trivial inertia groups, then the G -action on Y lifts over $W(k)$.*

(ii) *In fact, even if G acts on Y with prime-to- p inertia groups, then the G -action on Y lifts over $W(k)$. In particular, any group of prime-to- p order is an Oort group.*

Proof. In case (i), the G -cover $f : Y \rightarrow Y/G \cong X$ is étale. By Proposition 2.1, we can lift X to a flat, smooth curve $X_{W(k)}$. By Grothendieck's theory of étale lifting ([SGA1, I, Corollaire 8.4], combined with [SGA1, III, Prop. 7.2]), the G -cover f lifts to a G -cover $f_{W(k)} : Y_{W(k)} \rightarrow X_{W(k)}$ over $W(k)$. In case (ii), we instead use Grothendieck's theory of tame lifting ([SGA1, XIII, Corollaire 2.12], or [Wew99] for an exposition). \square

Remark 2.3. In §6, we will see how Proposition 2.2 also follows from the local-global principle.

Remark 2.4. In fact, if $f : Y \rightarrow X$ is tamely ramified (as in the proof of Proposition 2.2), then the generic fiber of $f_{W(k)}$ is branched at the same number of points with the same branching indices as f . Furthermore, the lifting of f over $W(k)$ (or over any finite $R/W(k)$) is unique once the branch points of the generic fiber are specified.

This has the following consequence about fundamental groups of curves: Let U be an affine curve over k , let X be the smooth projective closure of U , let $X_{W(k)}$ be a lift of X to $W(k)$, and let $U_{W(k)} \subseteq X_{W(k)}$ be a lift of U to $W(k)$ such that the generic fiber of $X_{W(k)} \setminus U_{W(k)}$ contains exactly one point specializing to each point of $X \setminus U$. Let L be an algebraic closure of $\text{Frac}(W(k))$ and let U_L and X_L be the base changes of $U_{W(k)}$ and $X_{W(k)}$ to L . If $\pi_1^t(U)$ is the *tame fundamental group* of U (that is, the automorphism group of the pro-universal tame cover of X , étale above U), and $\pi_1^t(U_L)$ is the *p -tame fundamental group* of U_L (same definition, but replace “tame” with “ramified of prime-to- p index”), then there is a natural surjection

$$\phi : \pi_1^t(U_L) \rightarrow \pi_1^t(U),$$

given as follows: If $\gamma \in \pi_1^t(U_L)$, then $\phi(\gamma)$ is determined by how it acts on tame covers $f : Y \rightarrow X$, étale over U . Take the unique lift $f_{W(k)}$ of f over $W(k)$ such that if $f_{\mathcal{O}_L}$ is the base change of $f_{W(k)}$ to the valuation ring \mathcal{O}_L of L , the branch locus of the generic fiber of $f_{\mathcal{O}_L}$ is contained in $X_L \setminus U_L$. Note that the special fiber of $f_{\mathcal{O}_L}$ is identical to that of $f_{W(k)}$. By definition, γ acts on the generic fiber of this base change of f , and the reduction of this action is the desired action on Y . It is also clear that ϕ is surjective, as one can compatibly lift any compatible system of automorphisms of tame covers of X , étale over U . Since $\pi_1^t(U_L)$ is well understood (as are all fundamental groups of curves over algebraically closed fields of characteristic zero), the existence of ϕ gives us useful information about $\pi_1^t(U)$. For instance, we obtain that $\pi_1^t(U)$ is topologically finitely generated.

Given that we can (essentially uniquely) lift tame covers of curves to characteristic zero, the next natural question is to ask when we can lift wild covers, and what kinds of moduli these lifts have. Hopefully, for affine curves U/k , this can shed some light on larger quotients of $\pi_1(U)$ than $\pi_1^t(U)$. In any case, this gives us motivation to study the lifting problem for a G -action on Y by examining the local properties of the G -cover $f : Y \rightarrow Y/G$ near its ramification points.

§3. Reduction to the local lifting problem

The *local-global principle* below allows us to study the lifting problem by way of the local lifting problem.

Theorem 3.1 (Local-global principle). *Let Y be a smooth, projective, connected curve over k , with a faithful action of G by k -automorphisms. Let $y_1, \dots, y_s \in Y$ be the points where G acts with nontrivial inertia. For each j , $1 \leq j \leq s$, let G_j be the inertia group of y_j in G , and let $\iota_j : G_j \hookrightarrow \text{Aut}_{k, \text{cont}} k[[u_j]]$ be the induced local action on the complete local ring of y_j . If $R/W(k)$ is finite, then the curve Y (with G -action) lifts over R iff each of the local G_j -actions lifts over R .*

Sketch of proof. (see [Sai10, §1.2], or [Gar96, §3] for many more details) Clearly, if the G -action on Y lifts over R , then so do all the local actions. Now, assume each of the local actions lifts over R . Consider the G -cover $f : Y \rightarrow X := Y/G$. By Proposition 2.1, X has a smooth lift X_R over R . Let \mathcal{X} be the formal completion of X_R at X . Let B be the branch locus of f , set $V = X \setminus B$, and set $W = f^{-1}(V) = Y \setminus \{y_1, \dots, y_s\}$. Let $\mathcal{V} \subseteq \mathcal{X}$ be the formal subscheme associated to $V \subseteq X$ (§A.4). By Grothendieck’s theory of étale lifting ([SGA1, I, Corollaire 8.4]), the G -cover $f|_W : W \rightarrow V$ lifts to an étale G -cover of formal schemes $\mathcal{W} \rightarrow \mathcal{V}$ over R . The boundary of \mathcal{W} is isomorphic to disjoint union $\coprod_{j=1}^s \mathcal{B}_j$, where each \mathcal{B}_j is isomorphic to the boundary of a disc (§A.1). Each \mathcal{B}_j corresponds to the point y_j , and G_j acts on \mathcal{B}_j .

By assumption, each local G_j -action on $\hat{\mathcal{O}}_{Y, y_j}$ lifts over R to a continuous R -linear G_j -action on the open disc $\mathcal{D}_j \cong \text{Spf } R[[U_j]]$. The action of G_j on \mathcal{D}_j induces an action on its boundary $\partial \mathcal{D}_j$. In fact, the G_j -action on $\partial \mathcal{D}_j$ is isomorphic to the action on \mathcal{B}_j . Thus by identifying \mathcal{B}_j and $\partial \mathcal{D}_j$, we can use formal patching to “glue” each of these discs \mathcal{D}_j to \mathcal{W} in a G_j -equivariant way. This yields a formal curve with G -action and projective special fiber (see, e.g., [Sai10, §1.2], or [Hen99, Ch. 3, §4] for more details). By Grothendieck’s Existence Theorem, this formal curve is the projective completion of a smooth projective curve Y_R with G -action. This is the lift we seek. \square

Remark 3.2. Bertin and Mézard gave an alternate proof of Theorem 3.1 that relies on deformation theory (see [BM00, §3]). There is also a proof by Green and Matignon ([GM98, III]).

Because we have Theorem 3.1 at our disposal, the rest of this paper will focus exclusively on the local lifting problem. For interesting work involving global lifting, see [Oor87], [OSS89], and [CGH08]. For an overview of some of the techniques used in [Oor87] and [OSS89], see [Bre05], especially Chapter 3.

§4. Generalities on local extensions

4.1. Power series rings

Consider the power series ring $A = k[[u]]$. A continuous k -automorphism γ of A (for the u -adic topology) is determined by the image of u . In particular, it is necessary and sufficient to have $\gamma(u) = \sum_{i=1}^{\infty} a_i u^i$, with $a_1 \neq 0$. A first, naïve approach to the local lifting problem would be to try to write down a group G of automorphisms of A explicitly as power series, and then try to lift them explicitly. However, this method is of limited usefulness. It quickly becomes difficult even to write down power series of finite order in $k[[u]]$, let alone to write down a lift to $R[[U]]$, where $R/W(k)$ is finite. For order m prime to p , the series $\gamma(u) = \zeta_m u$, with ζ_m a primitive m th root of unity, works. For order p , the series $\gamma(u) = \frac{u}{1+u} = u - u^2 + u^3 - \cdots$ works. But it is already not easy to see how to lift this to an order p automorphism of $R[[U]]$ (note that lifting the power series $\gamma(u)$ to $R[[U]]$ naïvely will not in general result in an automorphism of order p).

For automorphisms of order divisible by p^2 , the situation is even worse. In fact, the only explicit continuous k -automorphism of $k[[u]]$ of order p^n known to the author, for $n \geq 2$, is an automorphism γ of order 4 given in [CS10] by

$$\gamma(u) = u + u^2 + \sum_{j=0}^{\infty} \sum_{\ell=0}^{2^j-1} u^{6 \cdot 2^j + 2\ell}.$$

Trying to lift this automorphism while maintaining its order is a nightmare that we shall not attempt.

In order to lift a G -action on $k[[u]]$, we must not only lift automorphisms with the correct orders, but we must also take into account the group structure of G . Since even lifting the automorphisms with the correct orders seems to be beyond our reach, we will not pursue this method further.

4.2. Galois extensions

The more fruitful approach to the local lifting problem has been to view it from the perspective of Galois theory. This is analogous to studying the (global) lifting problem by studying branched covers of curves. Suppose the group G acts on $k[[u]]$ by continuous k -automorphisms. Then $k[[u]]^G$ is a complete DVR with residue field k , so by the structure theorem for complete DVRs ([Ser79, II, Theorem 2]), it must be abstractly isomorphic to $k[[t]]$. Choosing an isomorphism gives $k[[u]]/k[[t]]$ the structure of a G -extension. Conversely, if $A/k[[t]]$ is any G -Galois extension, then $A \cong k[[u]]$, again by the structure theorem. Likewise, if $R/W(k)$ is finite, then $R[[U]]^G \cong R[[T]]$ if G acts by continuous R -automorphisms ([Ray99, Proposition 2.3.1]), and any G -extension $A_R/R[[T]]$ for which $A_R \otimes_R k \cong k[[u]]$ must satisfy $A_R \cong R[[U]]$. We can now rephrase the local lifting problem as follows:

Question 4.1. [Local lifting problem, Galois formulation] Suppose G is a finite group, and $A/k[[t]]$ is a G -Galois extension, for a finite group G . Does there exist $R/W(k)$ finite, and a G -Galois extension $A_R/R[[T]]$ such that $A_R \otimes_R k \cong A$ and the G -action on A_R reduces to the given G -action on A ?

If such a lift exists, we say that $A_R/R[[T]]$ is a *lift* of $A/k[[t]]$ over R , or that $A/k[[t]]$ *lifts to characteristic zero* (or *lifts over R*).

4.2.1. Properties of Galois extensions of complete DVRs. The following facts are from [Ser79, IV]. Let $F = k((t))$. If L/F is a finite G -Galois extension of complete discrete valuation fields, then G must be of the form $P \rtimes \mathbb{Z}/m$, where P is a p -group and m is prime to p . The group G has a filtration $G = G_0 \supseteq G_i$ ($i \in \mathbb{R}_{\geq 0}$) for the lower numbering, and $G \supseteq G^i$ for the upper numbering ($i \in \mathbb{R}_{\geq 0}$). If $i \leq j$, then $G_i \supseteq G_j$ and $G^i \supseteq G^j$. The subgroup G_i (resp. G^i) is known as the *i th higher ramification group for the lower numbering* (resp. *the upper numbering*). The two filtrations are related by Herbrand's formula (see [Ser79, IV, §1, §3]).

One knows that $G_0 = G^0 = G$, and that $G_1 = G^{\frac{1}{m}} = P$. For sufficiently large i , $G_i = G^i = \{id\}$. Any i such that $G^i \supsetneq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an *upper jump* of the extension L/F . Likewise, if $G_i \supsetneq G_{i+\epsilon}$ for $\epsilon > 0$, then i is called a *lower jump* of L/F . If i is a lower (resp. upper) jump, $i > 0$, and $\epsilon > 0$ is sufficiently small, then $G_i/G_{i+\epsilon}$ (resp. $G^i/G^{i+\epsilon}$) is an elementary abelian p -group. The lower jumps are all integers.

The above discussion applies equally to G -extensions of $k[[t]]$. In particular, any group G that arises in the local lifting problem is of the

form $P \rtimes \mathbb{Z}/m$, where P is a p -group and $p \nmid m$. The degree δ of the different of a G -extension $A/k[[t]]$ is given by the formula

$$(4.1) \quad \delta = \sum_{i=0}^{\infty} (|G_i| - 1).$$

The fact that the different can be determined from the higher ramification filtration for the lower numbering (and, indeed, for the upper numbering also) will be important for determining whether a given extension of $R[[T]]$ for $R/W(k)$ finite is, in fact, a lift (see §4.3).

Remark 4.2. Note that if $A/k[[t]]$ is a wildly ramified G -extension, then the degree of the different of $A/k[[t]]$ is always strictly greater than $|G| - 1$. By the Hurwitz formula, this means that if $Y \rightarrow X$ is a wildly ramified cover of curves over k , then the genus of Y is always higher than it would be if the cover had the same ramification points and indices, but was in characteristic 0.

4.2.2. Cyclic Extensions. If $G \cong \mathbb{Z}/p^n$, then any G -extension must have n different upper jumps $u_1 < \cdots < u_n$.

Lemma 4.3. Set $u_0 = 0$. If L/K is a \mathbb{Z}/p^n -extension with upper jumps $u_1 < \cdots < u_n$, then the degree of the different of L/K is

$$p^n - 1 + \sum_{i=1}^n p^{i-1} (p^{n-i+1} - 1) (u_i - u_{i-1}).$$

Proof. This follows from (4.1) and Herbrand's formula. \square

4.3. Birational lifts and the different criterion

Usually, when dealing with Galois extensions of $k[[t]]$, it will be more convenient to deal with extensions of fraction fields than extensions of rings. For instance, by Artin-Schreier theory, one knows that any \mathbb{Z}/p -extension $L/k((t))$ is given by an equation of the form $y^p - y = f(t)$. But writing down equations for the integral closure of $k[[t]]$ in L is much more difficult. So we will often want to think of a Galois ring extension in terms of the associated extension of fraction fields. In particular, we have the following *birational local lifting problem*:

Question 4.4. [Birational local lifting problem] Let $A/k[[t]]$ be a G -extension. Does there exist $R/W(k)$ finite, and a G -extension $M/\text{Frac}(R[[T]])$ such that:

- (1) If A_R is the integral closure of $R[[T]]$ in M , then the integral closure of $A_R \otimes_R k$ is isomorphic to A ,

- (2) The G -action on $\text{Frac}(A) = \text{Frac}(A_R \otimes_R k)$ induced from that on A_R restricts to the given G -action on A ?

If so, we say that $A/k[[t]]$ *lifts birationally to characteristic zero* and that $A_R/R[[T]]$ is a *birational lift* of $A/k[[t]]$ to characteristic zero.

The birational local lifting problem evidently requires less than the local lifting problem to answer in the affirmative. In fact, Garuti has proved the following:

Theorem 4.5 ([Gar96]). *Any G -extension $A/k[[t]]$ lifts birationally to characteristic zero.*

Sketch of proof. If $A/k[[t]]$ is a G -extension, then by [Kat86, Theorem 1.4.1], there is a G -cover $f : Y \rightarrow X := \mathbb{P}_k^1$ such that f is totally ramified at a point $y \in Y$, and tamely ramified away from y . The extension of complete local rings associated to $y \mapsto f(y)$ is isomorphic to $A/k[[t]]$. Write $U = X \setminus \{f(y)\}$ and $V = Y \setminus \{y\}$. Let \mathcal{X} be the formal completion of $\mathbb{P}_{W(k)}^1$ at its special fiber X , let \mathcal{U} be the natural formal lift of U inside \mathcal{X} (see §A.4), and let \mathcal{D} be the formal open disc $\mathcal{X} \setminus \mathcal{U}$. Thus $\mathcal{D} \cong \text{Spf } W(k)[[T]]$.

By Grothendieck's theory of tame lifting ([SGA1, XIII, Corollaire 2.12]), the G -cover $f : V \rightarrow U$ lifts to a tame G -cover $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{U}$ of formal schemes. Applying [Gar96, Corollaire 2.14] shows that there exists $R/W(k)$ finite such that \mathcal{F} , when base changed to R , extends to a G -cover $\mathcal{Y} \rightarrow \mathcal{X}$ of normal formal schemes, where the special fiber of \mathcal{Y} is possibly singular. The equation of this G -cover over \mathcal{D} is a G -extension $A_R/R[[T]]$ that is a birational lift of $A/k[[t]]$. \square

Remark 4.6. Of course, most of the real work is contained in [Gar96, Corollaire 2.14]. For an overview of this corollary that uses less group theory than [Gar96], see [Bre05, Ch. 4], especially section 1. The argument given above is somewhat artificial, as it is not actually necessary to globalize the problem by building the branched cover f . However, in [Gar96], the viewpoint is global, so we globalize in order to cite it.

Remark 4.7. Saïdi has refined the arguments of [Gar96] to show that the birational local lifting problem is solvable in *towers*. More specifically, let $B/k[[t]]$ be a G -Galois extension, let Γ be a normal subgroup of G , and let $A = B^\Gamma$. If $A_R/R[[T]]$ is a birational lift of $A/k[[t]]$, then there is a finite extension R'/R and a birational lift $B_{R'}/R'[[T]]$ of $B/k[[t]]$ such that $(B_{R'})^\Gamma \cong A_R \otimes_R R'$. See [Sai10, Theorem 2.5.5], where birational lifts are called “Garuti lifts.”

The following criterion is extremely useful for seeing when a birational lift is actually a lift (i.e., when $A_R \otimes_R k$ is already integrally closed, thus isomorphic to A , in the language of Question 4.4).

Proposition 4.8 (The different criterion). *Suppose $A_R/R[[T]]$ is a birational lift of the G -Galois extension $A/k[[t]]$. Let $K = \text{Frac}(R)$, let δ_η be the degree of the different of $(A_R \otimes_R K)/(R[[T]] \otimes_R K)$, and let δ_s be the degree of the different of $A/k[[t]]$. Then $\delta_s \leq \delta_\eta$, and equality holds if and only if $A_R/R[[T]]$ is a lift of $A/k[[t]]$.*

Sketch of proof. (cf. [GM98, I, 3.4]) For any free module C/B of rank r , write $\det(C/B) = \bigwedge_B^r(C)$. Consider the trace map

$$\tau : \det(A_R/R[[T]]) \times \det(A_R/R[[T]]) \rightarrow R[[T]]$$

induced by the map $A_R \times A_R \rightarrow R[[T]]$ given by $(x, y) \mapsto \text{tr}_{A_R/R[[T]]}(xy)$. If π is a uniformizer of R , then by the Weierstrass Preparation Theorem, the image of τ in $R[[T]]$ is generated by $\pi^n P(T)$, with $n \geq 0$ and $P(T)$ a distinguished polynomial (i.e., P has a unit leading coefficient, and all other coefficients are in the maximal ideal of R). Tensoring the map τ with K shows that $\delta_\eta = \dim_{R[[T]] \otimes_R K}(\text{coker}(\tau \otimes_R K)) = \deg P$. But setting $\tau' = (\pi^{-n}\tau) \otimes_R k$, we see that the image of

$$\tau' : \det((A_R \otimes_R k)/k[[t]]) \times \det((A_R \otimes_R k)/k[[t]]) \rightarrow k[[t]]$$

is generated by the reduction of $P(T)$ to characteristic p , so $\dim_{k[[t]]}(\text{coker } \tau') = \deg P = \delta_\eta$.

Let $\tilde{\tau}$ be the trace map $\det(A/k[[t]]) \times \det(A/k[[t]]) \rightarrow k[[t]]$. Then $\dim_{k[[t]]}(\text{coker } \tilde{\tau}) = \delta_s$. Since A is the integral closure of $A_R \otimes_R k$, [Ser79, III, Proposition 5] shows that $\delta_s \leq \delta_\eta$, with equality if and only if $A_R \otimes_R k \cong A$. \square

Remark 4.9. In [Sai10], Saïdi introduces the idea of a *fake lifting* of a G -extension. A fake lifting of a G -extension $A/k[[t]]$ is a birational lift that is not an actual lift, but such that the degree of the different on the generic fiber is minimal among birational lifts. By Proposition 4.8, the extension $A/k[[t]]$ lifts to characteristic zero iff it has no fake liftings. Thus, one can try to answer a lifting problem in the affirmative by showing that the properties that fake liftings must satisfy are so restrictive that they cannot possibly simultaneously hold. In [Sai10], this is applied to give a proof that \mathbb{Z}/p is a local Oort group, and also to show that certain \mathbb{Z}/p^2 -extensions lift to characteristic zero (these results were already known—see §6).

§5. Obstructions to lifting

In §1.1, we discussed some obstructions to the (global) lifting problem. In this section, we will discuss the *Katz-Gabber-Bertin obstruction*, or *KGB obstruction* ([CGH09]), which is the most effective way of showing that a particular local lifting problem does not have a solution.

5.1. The KGB obstruction

Let $A/k[[t]]$ be a G -extension, where $G \cong P \rtimes \mathbb{Z}/m$, with P a p -group and $p \nmid m$. A theorem of Katz and Gabber ([Kat86, Theorem 1.4.1]) states that there exists a unique G -cover $Y \rightarrow \mathbb{P}_k^1$ that is étale outside $t \in \{0, \infty\}$, tamely ramified of index m above $t = \infty$, and totally ramified above $t = 0$ such that the extension of complete local rings at $t = 0$ is given by $A/k[[t]]$. This is called the *Katz-Gabber cover associated to $A/k[[t]]$* . By Theorem 3.1 and Proposition 6.1 (which does not depend on anything in this section), the G -cover $f : Y \rightarrow \mathbb{P}_k^1$ lifts to characteristic zero iff the extension $A/k[[t]]$ does.

Let $R/W(k)$ be finite, and let $K = \text{Frac}(R)$. Suppose $f_R : Y_R \rightarrow \mathbb{P}_R^1$ is a lift of f to characteristic zero. Since genus is constant in flat families, the genus of Y is equal to that of $Y_K := Y_R \times_R K$. Furthermore, if $H \leq G$ is any subgroup, then Y_R/H is a lift of Y/H with generic fiber Y_K/H , and the genus of Y/H is equal to that of Y_K/H .

Definition 5.1. Let $A/k[[t]]$ be a G -extension with associated Katz-Gabber G -cover $Y \rightarrow \mathbb{P}_k^1$. Then the *KGB obstruction vanishes* for $A/k[[t]]$ if there exists a G -cover $X \rightarrow \mathbb{P}^1$ over a field of characteristic zero such that, for each subgroup $H \subseteq G$, the genus of Y/H is equal to the genus of X/H . Equivalently (by the Hurwitz formula), the degree of the ramification divisor of $Y/H \rightarrow \mathbb{P}_k^1$ is equal to that of $X/H \rightarrow \mathbb{P}^1$ for each subgroup $H \subseteq G$. Equivalently, the degree of the ramification divisor of $Y \rightarrow Y/H$ is equal to that of $X \rightarrow X/H$ for each subgroup $H \subseteq G$.

Clearly, if $A/k[[t]]$ lifts to characteristic zero, its KGB obstruction must vanish. If $G \cong P \rtimes \mathbb{Z}/m$ as above and the KGB obstruction vanishes for *all* G -extensions $A/k[[t]]$, then G is called a *KGB group* for k .

The following classification of all the KGB groups is due to Chinburg, Guralnick, and Harbater.

Theorem 5.2 ([CGH09], Theorem 1.2). *The KGB groups for k consist of the cyclic groups, the dihedral group D_{p^n} for any n , the group A_4 (for $\text{char}(k) = 2$), and the generalized quaternion groups Q_{2^m} of order 2^m for $m \geq 4$ (for $\text{char}(k) = 2$).*

Remark 5.3. There is another obstruction, called the *Bertin obstruction* ([Ber98]), that was proven in [CGH09] to be strictly weaker than the KGB obstruction. Since it is also more difficult to describe, we will not discuss it here.

Since any local Oort group must be a KGB group, one can ask which of the KGB groups are local Oort groups. The generalized quaternion groups were shown *not* to be local Oort groups in [BrW09]. The obstruction developed in [BrW09] to show this is called the *Hurwitz tree obstruction*, and has to do with the non-existence of a kind of generalized Hurwitz tree (for basics on Hurwitz trees, see §7.3).

The group A_4 was announced to be a local Oort group in [BoW06]. The group D_p was shown to be a local Oort group by Bouw and Wewers ([BoW06]) for p odd and by Pagot ([Pag02]) for $p = 2$. We will discuss dihedral groups more in §7.

For cyclic groups, a major guiding problem in the field is the *Oort conjecture*.

Conjecture 5.4 (Oort Conjecture). *Any cyclic group is a local Oort group for any field k .*

If the cyclic group G has $v_p(|G|) = 0$ (resp. $v_p(|G|) = 1$, resp. $v_p(|G|) = 2$), then G is a local Oort group by Proposition 6.1 (resp. [OSS89], resp. [GM98]). These results, along with the Oort conjecture in general, will be discussed in detail in §6. We note that the Hurwitz tree obstruction to the Oort conjecture was shown to vanish in [Bre09, Ch. 4].

We also have the following strengthening of the Oort conjecture:

Conjecture 5.5 (Strong Oort Conjecture). *If G is a cyclic group, then any G -extension $A/k[[t]]$ lifts over $W(k)[\zeta_{|G|}]$, where $\zeta_{|G|}$ is a primitive $|G|$ th root of unity.*

In fact, the results in [OSS89] and [GM98] prove the strong Oort conjecture for $v_p(|G|) \leq 2$.

Remark 5.6. It seems plausible that all D_{p^n} might be local Oort groups for any k of characteristic p , but no case has been proven with $n > 1$ and no one has been willing to conjecture this. See Question 5.11.

Remark 5.7. The Oort conjecture was originally phrased in [Oor87, §7] as the global statement that “it seems reasonable to expect that [lifting] is possible for every automorphism of an algebraic curve.”

5.2. Non-KGB groups

For a group G that is not a KGB group, we can ask when a G -extension has vanishing KGB obstruction. In this section, we will focus

on two of the simplest cases, namely $G = \mathbb{Z}/p \times \mathbb{Z}/p$, for $p > 2$, and $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, when $p \nmid m$. We will use the fact that any n -tuple of generators (g_1, \dots, g_n) of G such that $g_1 \cdots g_n = id$ gives rise to a G -cover of \mathbb{P}_K^1 branched at n points for any algebraically closed field K of characteristic zero (see, e.g., [CH85]). The branching indices of the n branch points are the orders of g_1, \dots, g_n , respectively (in fact, the inertia groups above these points are conjugates of the cyclic groups generated by the g_i). The n -tuple (g_1, \dots, g_n) is called a *branch cycle description* of the G -cover.

Proposition 5.8. *Let m_1 and m_2 be the first and second lower jumps of a $\mathbb{Z}/p \times \mathbb{Z}/p$ -extension $A/k[[t]]$. If the KGB obstruction for $A/k[[t]]$ vanishes, then $m_1 \equiv -1 \pmod{p}$. The converse is true unless $p = 3$ and $m_1 = m_2 = 2$.*

Proof. (cf. [GM98, I, Theorem 5.1]) The degree of the ramification divisor of the Katz-Gabber cover $g : Y \rightarrow \mathbb{P}_k^1$ associated to $A/k[[t]]$ is the same as the degree of the different of $A/k[[t]]$, which is $\delta = (m_1 + 1)(p^2 - 1) + (m_2 - m_1)(p - 1)$ by (4.1). Now, each branch point of a $\mathbb{Z}/p \times \mathbb{Z}/p$ -cover in characteristic zero is branched of index p , and thus contributes $p(p - 1)$ to the degree of the ramification divisor. Thus, for the KGB obstruction to vanish, we must have $p(p - 1) \mid \delta$. This is equivalent to $p \mid m_2 + 1$. Since $m_1 \equiv m_2 \pmod{p}$ ([Ser79, IV, Prop. 11]), we have $m_1 \equiv -1 \pmod{p}$.

Conversely, suppose $m_1 \equiv -1 \pmod{p}$ (and thus $m_2 \equiv -1 \pmod{p}$ as well). Consider a branch cycle description of the form

$$(g_{0,1}, \dots, g_{0,r_0}, g_{1,1}, \dots, g_{1,r_2}, \dots, g_{p,1}, \dots, g_{p,r_p})$$

corresponding to branch points

$$(x_{0,1}, \dots, x_{0,r_0}, x_{1,1}, \dots, x_{1,r_2}, \dots, x_{p,1}, \dots, x_{p,r_p}),$$

where $r_0 = \dots = r_{p-1} = \frac{m_1+1}{p}$ and $r_p = \frac{m_2+1}{p}$. Write $G := \mathbb{Z}/p \times \mathbb{Z}/p$ additively, such that $(1, 0)$ is an element of G_{m_2} . Then we can take each $g_{i,j}$ to be a non-identity multiple of $(i, 1)$ for $0 \leq i \leq p - 1$, and each $g_{p,j}$ to be a non-identity multiple of $(1, 0)$. We leave it as an exercise to show that, unless $p = 3$ and $m_1 = m_2 = 2$, these choices can be made so that $\sum_{i,j} g_{i,j} = (0, 0)$. This gives rise to a G -cover $X \rightarrow \mathbb{P}_K^1$, with $\text{char}(K) = 0$.

Suppose H is a subgroup of G . We calculate the degrees of the ramification divisors \mathcal{R}_X of $X \rightarrow X/H$ and \mathcal{R}_Y of $Y \rightarrow Y/H$. If H is trivial, then these degrees are both zero. If $H = G$, then $|\mathcal{R}_Y| = (m_1 + 1)(p^2 - 1) + (m_2 - m_1)(p - 1)$ by the same calculation as earlier

in the proof, and it is easy to see that $|\mathcal{R}_X| = (m_1 + 1 + \frac{m_2+1}{p})(p^2 - p)$, which is equal to $|\mathcal{R}_Y|$.

Now assume H has order p . The H -cover $X \rightarrow X/H$ is ramified exactly at the ramification points above those $x_{i,j}$ for which $g_{i,j}$ generates H . Since p points of X lie above each $x_{i,j}$, we obtain $|\mathcal{R}_X| = (m_2 + 1)(p - 1)$ if H is generated by $(1, 0)$, and $(m_1 + 1)(p - 1)$ if not.

If the H_j are the higher ramification subgroups for the lower numbering for $Y \rightarrow Y/H$ at the ramified point, and the G_j are the higher ramification groups for $Y \rightarrow \mathbb{P}_k^1$, then $H_j = H \cap G_j$ ([Ser79, Prop. 2]). By (4.1), we have $|\mathcal{R}_Y| = \sum_{j \geq 0} |H_j - 1|$. Note that $H \cap G_j \cong \mathbb{Z}/p$ for $j \leq m_1$, and is trivial when $j > m_1$, except if H is generated by $(1, 0)$, in which case $H_j \cong \mathbb{Z}/p$ for $m_1 < j \leq m_2$. So one sees that $|\mathcal{R}_Y| = (m_2 + 1)(p - 1)$ if H is generated by $(1, 0)$, or $(m_1 + 1)(p - 1)$ if not. Thus, the G -cover $X \rightarrow \mathbb{P}^1$ causes the KGB obstruction to vanish. \square

Proposition 5.9. *Let $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where G is not cyclic and $p \nmid m$. Consider a G -extension $A/k[[t]]$ with first positive lower jump h . Then the KGB obstruction for $A/k[[t]]$ vanishes iff $h \equiv -1 \pmod{m}$. Furthermore, for this to happen, we must have that the conjugation action of \mathbb{Z}/m on \mathbb{Z}/p^n is faithful.*

Proof. (cf. [BoW06, Prop. 1.3]) Let $g : Y \rightarrow \mathbb{P}_k^1$ be the Katz-Gabber cover associated to $A/k[[t]]$. Let $f : X \rightarrow \mathbb{P}_K^1$ be a G -cover in characteristic zero. Now, $Y/(\mathbb{Z}/p^n) \rightarrow \mathbb{P}_k^1$ is a \mathbb{Z}/m -cover, branched at two points, so $Y/(\mathbb{Z}/p^n)$ must have genus 0. Thus, if f is to be a witness to the vanishing of the KGB obstruction, then the \mathbb{Z}/m -cover $X/(\mathbb{Z}/p^n) \rightarrow \mathbb{P}_K^1$ must have genus 0. Since any element of $PGL_2(K)$ of finite order is conjugate (after a possible extension of K) to a diagonal element when $\text{char}(K) = 0$, such an element acts on \mathbb{P}^1 with exactly two fixed points. Thus $X/(\mathbb{Z}/p^n) \rightarrow \mathbb{P}_K^1$ must be branched at two points with order m . Since G is not cyclic, any element of G with order divisible by m in fact has order m . So the G -cover f must have two branch points of index m , along with some branch points of p -power index. Consider $\tilde{f} : X/(\mathbb{Z}/p^{n-1}) \rightarrow \mathbb{P}_K^1$. The degree of the ramification divisor of this branched cover is $2(mp - p) + r(mp - m)$, where r is the number of branch points of f with index p^n (these points are branched of index p in \tilde{f}).

Now, the first positive upper jump for $A/k[[t]]$ is h/m , and since the upper numbering is preserved under quotients ([Ser79, Prop. 14]), the first positive upper jump for $A^{\mathbb{Z}/p^{n-1}}/k[[t]]$ is h/m . So the first lower jump for $A^{\mathbb{Z}/p^{n-1}}/k[[t]]$ is h . The degree of the different of this extension

is then $mp - 1 + h(p - 1)$, by (4.1). Thus the degree of the ramification divisor of $Y/(\mathbb{Z}/p^{n-1}) \rightarrow \mathbb{P}_k^1$ is $(mp - p) + mp - 1 + h(p - 1)$. Equating this to $2(mp - p) + r(mp - m)$ yields $h = -1 + rm$, so $h \equiv -1 \pmod{m}$. Furthermore, it is well-known (see, e.g., [Pri02, Lemma 1.4.1(iv)]) that $(h, m) = 1$ implies that the action of \mathbb{Z}/m on the p -Sylow subgroup of $G/(\mathbb{Z}/p^{n-1})$ is faithful. Thus the action of \mathbb{Z}/m on $\mathbb{Z}/p^n \subseteq G$ is faithful.

Conversely, suppose $h \equiv -1 \pmod{m}$ (so the action of \mathbb{Z}/m on \mathbb{Z}/p^n is faithful). Let the positive upper jumps for $A/k[[t]]$ be $u_1 = h/m, u_2, \dots, u_n$. By [OP10, Theorem 1.1], we have that $mu_1 \equiv \dots \equiv mu_n \equiv -1 \pmod{m}$. Consider a branch cycle description of the form

$$(\gamma_1, \gamma_2, g_{1,1}, \dots, g_{1,r_1}, g_{2,1}, \dots, g_{2,r_2}, \dots, g_{n,1}, \dots, g_{n,r_n})$$

corresponding to branch points

$$(z_1, z_2, x_{1,1}, \dots, x_{1,r_1}, x_{2,1}, \dots, x_{2,r_2}, \dots, x_{n,1}, \dots, x_{n,r_n}),$$

where $r_1 = u_1 + \frac{1}{m}$ and $r_i = u_i - u_{i-1}$ for $2 \leq i \leq n$. We choose γ_1 and γ_2 to be elements of G of order m , and for $1 \leq i \leq n$, we choose each $g_{i,j}$ to be an element of G of order p^{n-i+1} . By choosing γ_1 correctly, we can ensure that $\gamma_1 \gamma_2 \prod_{i,j} g_{i,j}$ is the identity. This gives rise to a G -cover $X \rightarrow \mathbb{P}_K^1$, with $\text{char}(K) = 0$.

Suppose H is a subgroup of G of order $p^{n'} m'$. We calculate the degrees of the ramification divisors \mathcal{R}_X of $X \rightarrow X/H$ and \mathcal{R}_Y of $Y \rightarrow Y/H$. We get a contribution of $p^{\min(n', n-i+1)} - 1$ to $|\mathcal{R}_X|$ at each of the mp^{i-1} points of X lying above an $x_{i,j}$, for any j . We get a contribution of $m' - 1$ to $|\mathcal{R}_X|$ at each point above z_1 or z_2 whose inertia group intersects H nontrivially. The p^n subgroups of G of order m are all conjugate and disjoint except for the identity element, and each one is the inertia group at one point above z_1 and one point above z_2 . Furthermore, H contains $p^{n'}$ subgroups of order m' , each one contained in a unique subgroup of order m of G . So the total contribution to $|\mathcal{R}_X|$ for the points above z_1 and z_2 is $2p^{n'}(m' - 1)$. We have shown:

$$|\mathcal{R}_X| = 2p^{n'}(m' - 1) + \sum_{i=1}^n r_i mp^{i-1} (p^{\min(n', n-i+1)} - 1).$$

In characteristic p , the tamely ramified point of $Y \rightarrow \mathbb{P}_k^1$ contributes $p^{n'}(m' - 1)$ to $|\mathcal{R}_Y|$ for the same reasons as in the last paragraph. Using Herbrand's formula, the higher ramification filtration of $Y \rightarrow \mathbb{P}_k^1$ at the wildly ramified point for the lower numbering has one group of order $p^n m$, and $mp^{i-1}(u_i - u_{i-1})$ groups of order p^{n-i+1} for $1 \leq i \leq n$, where we set $u_0 = 0$. If the H_j are the higher ramification subgroups

for $Y \rightarrow Y/H$ at this point, and the G_j are the higher ramification groups for $Y \rightarrow \mathbb{P}_k^1$, then $H_j = H \cap G_j$ ([Ser79, Prop. 2]). By (4.1), the contribution to $|\mathcal{R}_Y|$ at this point is $\sum_{j \geq 0} |H_j - 1|$. Note that if $j > 0$, then $|H_j|$ is the smaller of the order of the p -Sylow subgroups of H and G_j . We conclude that

$$|\mathcal{R}_Y| = p^{n'}(m' - 1) + p^{n'}m' - 1 + \sum_{i=1}^n mp^{i-1}(u_i - u_{i-1})(p^{\min(n', n-i+1)} - 1).$$

Some rearrangement shows that $|\mathcal{R}_X| = |\mathcal{R}_Y|$. Thus the KGB obstruction vanishes. \square

Remark 5.10. If p is odd, then any action of the dihedral group D_{p^n} satisfies the condition of Proposition 5.9 ([Pri02, Lemma 1.4.1] or [OP10, Theorem 1.1]). So Proposition 5.9 shows that D_{p^n} is a KGB group for p odd.

Question 5.11. Does every $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ -action satisfying the condition of Proposition 5.9 lift to characteristic zero?

Question 5.11 has a positive answer for $n = 1$ (Theorem 7.1). For $n > 1$, however, I am not aware of a single G -extension in this form that is known either to lift or not to lift. But there are D_4 -extensions that are known to lift from characteristic 2 to characteristic zero ([Bre08]).

5.3. Weak Oort groups

A group G is called a *weak Oort group* for k if there exists a G -extension $A/k[[t]]$ that lifts to characteristic zero. For instance, we mentioned above that D_4 is a weak Oort group for any algebraically closed field k of characteristic 2. Clearly, any local Oort group is a weak Oort group. We will content ourselves here to state two results about weak Oort groups, one positive and one negative. The first is due to Matignon:

Proposition 5.12 ([Mat99]). *The group $(\mathbb{Z}/p)^n$ is a weak Oort group for all primes p and positive integers n .*

Note that, if $n > 1$ and p is odd, then we can build a $(\mathbb{Z}/p)^n$ -extension with a $(\mathbb{Z}/p)^2$ -subextension that has nonvanishing KGB obstruction (Proposition 5.8). Thus $(\mathbb{Z}/p)^n$ is not an Oort group (or even a KGB group) by Theorem 5.2 (one also sees it is not an Oort group from the example in §1.1). For $p = 2$, we have that $(\mathbb{Z}/p)^n$ is not an Oort group for $n > 2$ by the example in §1.1.

The second result follows from [CGH09, Theorem 1.8].

Proposition 5.13. *If G contains an abelian subgroup that is neither cyclic nor a p -group, then G is not a weak Oort group.*

§6. Cyclic groups

Throughout §6, we take R to be a large enough finite extension of $W(k)$, and $K = \text{Frac}(R)$. Let $\mathcal{D} = \text{Spec } R[[T]]$ be the open unit disc. If L is any field of characteristic p , we say that $x \in L$ is a \wp *th power* if the equation $y^p - y = x$ has a solution in L . In this case y is called a \wp *th root* of x .

The purpose of §6 is to discuss progress toward the Oort conjecture (Conjecture 5.4).

6.1. Reduction to case of cyclic p -groups

Cyclic groups of prime-to- p order are local Oort groups:

Proposition 6.1. *Suppose x is any K -point of \mathcal{D} . If $A/k[[t]]$ is a \mathbb{Z}/m -extension, with $p \nmid m$, then it lifts to a \mathbb{Z}/m -extension $A_R/R[[T]]$. We can choose this lift so that if $f : \text{Spec } A_R \rightarrow \mathcal{D}$ is the induced map on schemes, then the generic fiber of f is branched only at x . If no branching behavior is specified, we can take $R = W(k)$.*

Proof. By Kummer theory, we may assume without loss of generality that A is given by $k[[t]][y]/(y^m - t)$. Think of x as an element of R with $v(x) > 0$ (see §A.1). Then choose A_R to be given by $R[[T]][Y]/(Y^m - (T - x))$. The reduction of A_R is as desired. If x is not specified, then the lift $R[[T]][Y]/(Y^m - T)$ is defined over $W(k)$. \square

Remark 6.2. Proposition 6.1, along with the local-global principle (Theorem 3.1), gives another proof of Proposition 2.2.

Using Proposition 6.1, we can reduce the local lifting problem for cyclic groups to the p -group case:

Proposition 6.3. *Let $G \cong \mathbb{Z}/mp^n$, where $p \nmid m$. If \mathbb{Z}/p^n is a local Oort group, then G is a local Oort group.*

Proof. Let H be the unique subgroup of G of order m and let H' be the unique subgroup of order p^n . Given a G -cover $f : Y \rightarrow \text{Spec } k[[t]]$, let $g : X \rightarrow \text{Spec } k[[t]]$ (resp. $g' : X' \rightarrow \text{Spec } k[[t]]$) be the unique quotient \mathbb{Z}/m -extension (resp. \mathbb{Z}/p^n -extension). Then the normalization of $X \times_{\text{Spec } k[[t]]} X'$ is isomorphic to Y . By assumption, g' lifts to a \mathbb{Z}/p^n -cover $g'_R : X'_R \rightarrow \mathcal{D}$, with generic fiber g'_K . By Proposition 6.1, g lifts to a \mathbb{Z}/m -cover $g_R : X_R \rightarrow \mathcal{D}$, and we can choose g so that the unique branch point of the generic fiber g_K of g_R is branched of index p^n in the generic fiber g'_K of g' .

Let X''_R be the normalization of $X_R \times_{\mathcal{D}} X'_R$. Then the canonical map $f_R : X''_R \rightarrow \mathcal{D}$ is clearly a birational lift of f . The degree of the different of g (and of g_K) is $m - 1$. Let δ be the degree of the different of g' (and

of g'_K). Using our assumptions on the branch loci of g_K and g'_K , one calculates the degree of the different of the generic fiber f_K of f_R to be $m\delta + m - 1$. On the other hand, the higher ramification groups $(H')^i$ for the upper numbering at the unique ramification point of g are the same as the corresponding groups G^i for f , as long as $i > 0$ ([Ser79, IV, Prop. 14]). Also, $|(H')^0| = p^n$ and $G^0 = mp^n$. Applying Herbrand's formula shows that $|H'_0| = p^n$, $|G_0| = mp^n$, and $|H'_i| = G_{mi}$ for $i \geq 1$. Then (4.1) shows that the degree of the different of f is also $m\delta + (m - 1)$. By Proposition 4.8, we conclude that f_R is a lift of f . \square

Remark 6.4. For any $i \in \mathbb{N}$, let ζ_i be a primitive i th root of unity. One would like to claim that if the strong Oort conjecture (Conjecture 5.5) holds for \mathbb{Z}/p^n , then it also holds for \mathbb{Z}/mp^n , for any m with $p \nmid m$. However, if the lift of g' in Proposition 6.3 can be done over $W(k)[\zeta_{p^n}]$, then the lift of f in Proposition 6.3 can be done over $W(k)[\zeta_{p^n}, \zeta_m] = W(k)[\zeta_{mp^n}]$ only if g'_K has a branch point of index p^n defined over $\text{Frac}(W(k)[\zeta_{mp^n}])$. It does not seem clear that this must be the case.

6.2. \mathbb{Z}/p^n -extensions in characteristic p

In attempting to prove the Oort conjecture, it is useful to have a somewhat explicit form for all \mathbb{Z}/p^n -extensions of $k((t))$. It is well-known, by Artin-Schreier theory, that any \mathbb{Z}/p -extension of $k((t))$ is given by an equation $y^p - y = f(t)$, where $f(t)$ is not a p th power in $k((t))$. It is clear that $f(t)$ must have some term with negative degree, as otherwise, letting $\tilde{f} = f - f(0)$, we have that $a - \tilde{f} - \tilde{f}^p - \tilde{f}^{p^2} - \dots$ is a p th root of f , where a is a p th root of $f(0)$. In fact, after a change of variable in t , we may assume $f(t) = t^{-j}$, and in this case j is the unique upper jump in the higher ramification filtration of the extension (see Proposition 6.5 below).

It is less well-known that there is an explicit form for all \mathbb{Z}/p^n -extensions of $k((t))$, written as successive Artin-Schreier extensions. The higher ramification filtration (and thus the degree of the different) can also be read off from this form.

Proposition 6.5. *There exist explicit polynomials f_1, \dots, f_{n-1} over \mathbb{F}_p in $1, \dots, n-1$ variables respectively, such that for any given \mathbb{Z}/p^n -extension $L/k((t))$, there is a choice of $x_1, \dots, x_n \in k((t))$ such that $L/k((t))$ is given (possibly after a change of variable in t) by adjoining*

y_1, \dots, y_n satisfying

$$\begin{aligned} y_1^p - y_1 &= x_1 \\ y_2^p - y_2 &= f_1(y_1) + x_2 \\ &\vdots \\ y_n^p - y_n &= f_{n-1}(y_1, \dots, y_{n-1}) + x_n. \end{aligned}$$

Furthermore, we can choose $x_1 = t^{-j}$, for some $j \geq 1$ not divisible by p , and we can choose x_2, \dots, x_n to be polynomials in t^{-1} with no terms of degree divisible by p . Conversely, every choice of x_1, \dots, x_n in this form gives a distinct \mathbb{Z}/p^n -extension. If (u_1, \dots, u_n) are the upper jumps of $L/k((t))$, then $u_1 = j$, and for $i > 1$, we have $u_i = \max(\deg(x_i), pu_{i-1})$.

We can pick a generator σ of \mathbb{Z}/p^n such that $\sigma^{p^{i-1}} y_i = y_i + 1$ for $1 \leq i \leq n$.

Proof. That the equations and action of σ can be chosen in this form is originally due to Schmid ([Sch36]), see also [OP10, §3]. The formula for the upper jumps was proven by Garuti ([Gar02, Theorem 1.1]). \square

Proposition 6.5 has the immediate corollary:

Corollary 6.6. *The upper jumps (u_1, \dots, u_n) of the extension above satisfy $u_i \geq pu_{i-1}$ for $2 \leq i \leq n$. If $u_i > pu_{i-1}$, then $p \nmid u_i$.*

Remark 6.7. The Artin-Schreier-Witt theory (detailed in [OP10, §3] and [Lan02, pp. 330-331]) says that for any field L of characteristic p , there is an association between truncated Witt vectors $x \in W_n(L)$ of length n and \mathbb{Z}/p^n -extensions of L . Furthermore, $x, y \in W_n(L)$ are associated to the same extension iff there exists $z \in W_n(L)$ with $x - y = F(z) - z$, where $F : W_n(L) \rightarrow W_n(L)$ is the Frobenius map. If $L = k((t))$, then the length n Witt vector $(x_1, \dots, x_n) \in W_n(L)$ is associated to the extension in 6.5.

6.3. Lifting \mathbb{Z}/p -extensions

The following proposition solves the local lifting problem for \mathbb{Z}/p -extensions.

Theorem 6.8. *The group \mathbb{Z}/p is a local Oort group. In particular, any G -extension $A/k[[t]]$ lifts over $W(k)[\zeta_p]$, where ζ_p is a primitive p th root of unity.*

Proof. By Proposition 6.5, A is the normalization of $k[[t]]$ in $L/k((t))$, where $L/k((t))$ is given by $y^p - y = t^{-u_1}$, and u_1 is the upper jump. Let

$R = W(k)[\zeta_p]$, and let $\lambda = \zeta_p - 1$. Then $v(\lambda^{p-1} + p) > 1$. Consider the integral closure A_R of $R[[T]]$ in the Kummer extension of $\text{Frac}(R[[T]])$ given by

$$Z^p = 1 + \lambda^p T^{-u_1}.$$

Making the substitution $Z = 1 + \lambda Y$, we obtain

$$(\lambda Y)^p + p\lambda Y + o(p^{p/(p-1)}) = \lambda^p T^{-u_1},$$

where $o(p^{p/(p-1)})$ represents terms with coefficients of valuation greater than $\frac{p}{p-1}$. This reduces to $y^p - y = t^{-u_1}$. So we have constructed a birational lift.

The degree of the different of $A/k[[t]]$ is $(u_1 + 1)(p - 1)$, by Lemma 4.3. On the other hand, the generic fiber of $\text{Spec } A_R \rightarrow \text{Spec } R[[T]]$ is branched at exactly $u_1 + 1$ points in the unit disc ($T = 0$ and T equals each u_1 th root of $-\lambda^p$). Since the ramification is tame, the degree of the different of $A/R[[T]]$ is $(u_1 + 1)(p - 1)$ as well. By Proposition 4.8, our birational lift is an actual lift. \square

Remark 6.9. Theorem 6.8, along with Proposition 6.3 and the local-global principle, shows that, for $G \cong \mathbb{Z}/pm$ with $p \nmid m$, all G -actions on curves lift to characteristic zero. The original proof of this is due to Oort-Sekiguchi-Suwa ([OSS89]), and is global in nature. In particular, the proof relies on intricate calculations involving extensions of group schemes and deformations of generalized Jacobians. The local proof above, due to Green and Matignon ([GM98]), is much simpler, once one admits the local-global principle.

6.4. Sekiguchi-Suwa Theory

Underlying the calculation in the proof of Theorem 6.8 is the so-called “Kummer-Artin-Schreier theory” in degree p . Its generalization to degree p^n by Sekiguchi and Suwa ([SS94], [SS99]) is called the “Kummer-Artin-Schreier-Witt theory.” While a full accounting of the theory is well beyond the scope of this paper, we will give a brief exposition below. For a more detailed exposition (although still not as detailed as the papers of Sekiguchi and Suwa), see [MRT11].

6.4.1. Kummer-Artin-Schreier theory. Phrased in the language of group schemes, Kummer theory (of degree p) studies torsors under the group scheme μ_p , whereas Artin-Schreier theory studies torsors under the group scheme \mathbb{Z}/p in characteristic p . Now, if $A/k[[t]]$ is a \mathbb{Z}/p -extension with a lift $A_R/R[[T]]$, then $A_R/R[[T]]$ is a torsor under a finite, flat group scheme over R whose generic fiber is μ_p and whose special fiber is \mathbb{Z}/p (of course, if R contains the p th roots of unity, then

$\mu_p \cong \mathbb{Z}/p$, but it is more natural to think of the generic fiber as μ_p). These are rank p subgroup schemes of \mathbb{G}_m and \mathbb{G}_a , respectively. So in order to understand lifts of \mathbb{Z}/p -extensions, one approach is to study group schemes over R whose generic fiber is \mathbb{G}_m and whose special fiber is \mathbb{G}_a .

For any $\mu \in \mathfrak{m}$, the maximal ideal of R , consider the group scheme

$$\mathcal{G}^{(\mu)} := \operatorname{Spec}(R[x, \frac{1}{1 + \mu x}])$$

with comultiplication law

$$(x, y) \mapsto \mu xy + x + y,$$

coinverse

$$x \mapsto -\frac{x}{1 + \mu x},$$

and counit $x \mapsto 0$. There is a map from $\mathcal{G}^{(\mu)} \rightarrow \mathbb{G}_m := R[x, \frac{1}{x}]$ given by $x \mapsto 1 + \mu x$, which is an isomorphism on the generic fiber. Indeed, a good way to think about the R -points of $\mathcal{G}^{(\mu)}$ is as elements of R with multiplication defined by $x * y = z$, where z is such that $1 + \mu z = (1 + \mu x)(1 + \mu y)$. Clearly, the special fiber of $\mathcal{G}^{(\mu)}$ is isomorphic to \mathbb{G}_a .

Suppose R contains the p th roots of unity. Let $\mu = \lambda$ as in the proof of Theorem 6.8, so $\mu = \zeta_p - 1$, where ζ_p is a nontrivial p th root of unity. Then there is a group scheme morphism $\psi : \mathcal{G}^{(\lambda)} \rightarrow \mathcal{G}^{(\lambda^p)}$ given by

$$(6.1) \quad x \mapsto \frac{(1 + \lambda x)^p - 1}{\lambda^p},$$

and this map is surjective in the étale topology. Note that ψ is the map that makes the following diagram commute, where the vertical arrows are the maps to \mathbb{G}_m in the previous paragraph:

$$\begin{array}{ccc} \mathcal{G}^{(\lambda)} & \xrightarrow{\psi} & \mathcal{G}^{(\lambda^p)} \\ \downarrow & & \downarrow \\ \mathbb{G}_m & \xrightarrow{(\cdot)^p} & \mathbb{G}_m. \end{array}$$

So the generic fiber of ψ is isomorphic to the p th power map on $\mathbb{G}_{m,K}$ and the special fiber of ψ , after a similar calculation to the proof of Theorem 6.8, is the map $x \mapsto x^p - x$ on $\mathbb{G}_{a,k}$. Since R contains the p th roots of unity, the kernel of ψ is the constant group scheme \mathbb{Z}/p . Thus we have an exact sequence of étale sheaves over R :

$$(6.2) \quad 0 \rightarrow \mathbb{Z}/p \rightarrow \mathcal{G}^{(\lambda)} \rightarrow \mathcal{G}^{(\lambda^p)} \rightarrow 0.$$

The generic fiber of (6.2) is just the Kummer exact sequence over K , whereas the special fiber is just the Artin-Schreier sequence over k .

Proposition 6.10. *Suppose R contains the p th roots of unity. If B is a flat local R -algebra, then any unramified \mathbb{Z}/p -cover $\mathrm{Spec} C \rightarrow \mathrm{Spec} B$ is given by a Cartesian diagram of the form*

$$\begin{array}{ccc} \mathrm{Spec} C & \longrightarrow & \mathcal{G}^{(\lambda)} \\ \downarrow & & \downarrow \psi \\ \mathrm{Spec} B & \longrightarrow & \mathcal{G}^{(\lambda^p)} \end{array}$$

That is, $C \cong B[Y]/(\frac{(1+\lambda Y)^p - 1}{\lambda^p} - X)$, with $X \in B$.

Proof. The short exact sequence (6.2) gives an exact sequence

$$0 \rightarrow \mathcal{G}^{(\lambda^p)}(B)/\mathcal{G}^{(\lambda)}(B) \rightarrow H^1(\mathrm{Spec} B, \mathbb{Z}/p) \rightarrow H^1(\mathrm{Spec} B, \mathcal{G}^{(\lambda)}).$$

But $H^1(\mathrm{Spec} B, \mathcal{G}^{(\lambda)}) = 0$ ([SS94, Theorem 2.2]), so \mathbb{Z}/p -torsors over $\mathrm{Spec} B$ are in one-to-one correspondence with elements $\mathcal{G}^{(\lambda^p)}(B)/\mathcal{G}^{(\lambda)}(B)$. Such elements can be interpreted exactly as elements $X \in B$, where X_1 is considered equivalent to X_2 if $(1 + \lambda^p X_1)/(1 + \lambda^p X_2) = (1 + \lambda Y)^p$ for some Y in B . The proposition follows by the standard expression of the coboundary map from degree 0 to degree 1 in étale cohomology. \square

Remark 6.11. If π is a uniformizer of R , and $B = R[[T]]_{(\pi)}$, then B is a local R -algebra with residue field $k((t))$. The lift exhibited in Theorem 6.8 comes (birationally) from choosing $X = T^{-u_1}$ in Proposition 6.10. So in some sense, we can say that our lift of a \mathbb{Z}/p -extension comes from the Kummer-Artin-Schreier theory. In §6.5, we will discuss how lifts of \mathbb{Z}/p^2 -extensions come from the Kummer-Artin-Schreier-Witt theory described below.

6.4.2. Kummer-Artin-Schreier-Witt theory. The point of Kummer-Artin-Schreier-Witt theory (or *Sekiguchi-Suwa theory*) is to generalize the exact sequence (6.2) to a sequence of group schemes with kernel \mathbb{Z}/p^n , and to generalize Proposition 6.10 to \mathbb{Z}/p^n -covers. In this section, we will concentrate on the theory itself, and in §6.5, we will show how this is applied to the lifting problem.

Assume that R contains the p^n th roots of unity. We seek an exact sequence of group schemes over R whose generic fiber is the Kummer-like sequence

$$(6.3) \quad 1 \rightarrow \mu_{p^n} \cong \mathbb{Z}/p^n \rightarrow (\mathbb{G}_m)^n \xrightarrow{\phi_n} (\mathbb{G}_m)^n \rightarrow 1,$$

where the surjection ϕ_n is given by

$$(x_1, \dots, x_n) \mapsto (x_1^p, x_2^p/x_1, x_3^p/x_2, \dots, x_n^p/x_{n-1}),$$

and whose special fiber is the *Artin-Schreier-Witt* sequence

$$(6.4) \quad 1 \rightarrow \mathbb{Z}/p^n \rightarrow W_n \xrightarrow{\varphi} W_n.$$

Here W_n is the scheme of length n Witt vectors over k , which is an n -fold extension of \mathbb{G}_a 's, and φ is the map $F - Id$, where F is the Frobenius map. In the Kummer-like sequence, since R contains the p^n th roots of unity, we have $\mu_{p^n} \cong \mathbb{Z}/p^n$, and we can think of the injection as sending 1 to $(\zeta_p, \zeta_{p^2}, \dots, \zeta_{p^n})$, where each ζ_{p^i} is a p^i th root of unity and $\zeta_{p^j}^{p^{j-i}} = \zeta_{p^i}$ for $j \geq i$.

Proposition 6.12 ([SS94], Theorem 7.1 or [SS99], Theorem 8.1). *For each positive integer n , there exists a flat group scheme \mathcal{W}_n over R (called a Kummer-Artin-Schreier-Witt group scheme) that fits into the exact sequence*

$$(6.5) \quad 0 \rightarrow \mathbb{Z}/p^n \rightarrow \mathcal{W}_n \xrightarrow{\psi_{\mathcal{W}}} \mathcal{V}_n := \mathcal{W}_n/(\mathbb{Z}/p^n) \rightarrow 0$$

of group schemes over R . The special fiber of (6.5) is isomorphic to the Artin-Schreier-Witt exact sequence (6.4) over k , and the generic fiber is isomorphic to the Kummer-like exact sequence (6.3) over K . Furthermore, \mathcal{W}_n is an n -fold extension of $\mathcal{G}^{(\lambda)}$'s, and as a scheme is given by

$$\mathrm{Spec} R[Y_1, \dots, Y_n, \frac{1}{1 + \lambda Y_1}, \frac{1}{F_1(Y_1) + \lambda Y_2}, \dots, \frac{1}{F_{n-1}(Y_1, \dots, Y_{n-1}) + \lambda Y_n}],$$

for explicitly determined polynomials F_1, \dots, F_{n-1} . Also, \mathcal{V}_n is an n -fold extension of $\mathcal{G}^{(\lambda^p)}$'s, and as a scheme is given by

$$\mathrm{Spec} R[X_1, \dots, X_n, \frac{1}{1 + \lambda^p X_1}, \frac{1}{G_1(X_1) + \lambda^p X_2}, \dots, \frac{1}{G_{n-1}(X_1, \dots, X_{n-1}) + \lambda^p X_n}],$$

for explicitly determined polynomials G_1, \dots, G_{n-1} .

Remark 6.13. While there is an explicit algorithm to calculate the polynomials F_i and G_i , the calculation gets extremely complicated as n gets large. Indeed, for $n = 3$, determining G_2 in [SS99] takes five pages. Likewise, the proof of Proposition 6.12 requires extraordinarily complicated calculations.

Remark 6.14. There are morphisms $\mathcal{W}_n \rightarrow (\mathbb{G}_m)^n$ and $\mathcal{V}_n \rightarrow (\mathbb{G}_m)^n$, given by

$$\alpha^{(n)} : (Y_1, \dots, Y_n) \mapsto (1 + \lambda Y_1, F_1(Y_1) + \lambda Y_2, \dots, F_{n-1}(Y_1, \dots, Y_{n-1}) + \lambda Y_n)$$

and

$$\beta^{(n)} : (X_1, \dots, X_n) \mapsto (1 + \lambda^p X_1, G_1(X_1) + \lambda^p X_2, \dots, G_{n-1}(X_1, \dots, X_{n-1}) + \lambda^p X_n),$$

respectively. These maps are isomorphisms on the generic fibers, and the diagram

$$(6.6) \quad \begin{array}{ccc} \mathrm{Spec} \mathcal{W}_n & \xrightarrow{\alpha^{(n)}} & (\mathbb{G}_m)^n \\ \downarrow \psi_n & & \downarrow \phi_n \\ \mathrm{Spec} \mathcal{V}_n & \xrightarrow{\beta^{(n)}} & (\mathbb{G}_m)^n \end{array}$$

commutes, where ϕ_n is given by (6.3) and ψ_n is given by (6.5).

We state the generalization of Proposition 6.10:

Proposition 6.15 ([SS94], Theorem 3.8 or [SS99], Theorem 3.8). *Suppose R contains the p^n th roots of unity. If B is a flat local R -algebra, then any unramified \mathbb{Z}/p^n -cover $\mathrm{Spec} C \rightarrow \mathrm{Spec} B$ is given by a Cartesian diagram of the form*

$$\begin{array}{ccc} \mathrm{Spec} C & \longrightarrow & \mathcal{W}_n \\ \downarrow & & \downarrow \psi_n \\ \mathrm{Spec} B & \longrightarrow & \mathcal{V}_n = \mathcal{W}_n/(\mathbb{Z}/p^n) \end{array}$$

Proof. One shows that $H^1(\mathrm{Spec} B, \mathcal{W}_n) = 0$ ([SS94, Theorem 3.6]), and then the proof is the same as that of Proposition 6.10. \square

6.5. Lifting \mathbb{Z}/p^2 -extensions

The aim of this section is to sketch the proof of the following theorem of Green and Matignon:

Theorem 6.16 ([GM98], Theorem 2). *The group \mathbb{Z}/p^2 is a local Oort group. Moreover, any \mathbb{Z}/p^2 -extension of $k[[t]]$ lifts over $W(k)[\zeta_{p^2}]$.*

Green and Matignon do this by making the diagram in Proposition 6.15 more explicit in the case $n = 2$. More specifically, they take $R = W(k)[\zeta_{p^2}]$ and $B = R[[T]]_{(\pi)}$, where π is a uniformizer of R , and write down a specific morphism $\mathrm{Spec} B \rightarrow \mathcal{V}_2$. By Proposition 6.15, this

gives an unramified \mathbb{Z}/p^2 -cover $\text{Spec } C \rightarrow \text{Spec } B$. They then show that $C'/R[[T]]$, where C' is the integral closure of $R[[T]]$ in $\text{Frac}(C)$, reduces to a specific \mathbb{Z}/p^2 -Galois extension $A/k[[t]]$ (in fact, it is an extension in the form of Proposition 6.5 where $x_2 = 0$). Thus $C'/R[[T]]$ is a lift of $A/k[[t]]$. Lastly, they show how, given a lift of $A/k[[t]]$, one can lift all other \mathbb{Z}/p^2 -extensions of $k[[t]]$.

We start by constructing the map $\text{Spec } C \rightarrow \text{Spec } B$. Let $\lambda = \zeta_p - 1$ where ζ_p is a nontrivial p th root of unity, let $\pi = \zeta_{p^2} - 1$, where $\zeta_{p^2}^p = \zeta_p$, and let $\mu = \pi - \pi^2/2 + \cdots + (-1)^p \pi^{p-1}/(p-1)$. Note that μ and π are both uniformizers of R . Recall that

$$\mathcal{W}_n \cong \text{Spec } R[Y_1, Y_2, \frac{1}{1 + \lambda Y_1}, \frac{1}{F_1(Y_1) + \lambda Y_2}]$$

and

$$\mathcal{V}_n \cong \text{Spec } R[X_1, X_2, \frac{1}{1 + \lambda^p X_1}, \frac{1}{G_1(X_1) + \lambda^p X_2}].$$

We have $F_1(Y_1) = \exp_p(\mu Y_1)$ and $G_1(X_1) = \exp_p(\mu^p X_1)$, where \exp_p is the truncated exponential including terms up through degree $p-1$. The formula for F_1 comes from [SS94], and Green and Matignon derive the formula for G_1 , although it can also be determined from [SS99]. Moreover, by [SS94, Theorem 7.1], there is a commutative diagram of group schemes over R :

$$\begin{array}{ccccc} \mathcal{W}_1 = \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{W}_2 & \xrightarrow{\alpha^{(2)}} & (\mathbb{G}_m)^2 \\ \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \phi_2 \\ \mathcal{V}_1 = \mathcal{G}^{(\lambda^p)} & \longrightarrow & \mathcal{V}_2 & \xrightarrow{\beta^{(2)}} & (\mathbb{G}_m)^2, \end{array}$$

where the ψ_i 's are from Proposition 6.12, the left horizontal arrows are given by $X_2 = 0$ and $Y_2 = 0$, the map ϕ_2 is from (6.3), the map $\alpha^{(2)}$ is given by $(Y_1, Y_2) \mapsto (1 + \lambda Y_1, F_1(Y_1) + \lambda Y_2)$, and $\beta^{(2)}$ is given by $(X_1, X_2) \mapsto (1 + \lambda^p X_1, G_1(X_1) + \lambda^p X_2)$ (the right hand square is nothing but (6.6), for $n = 2$). From this diagram, we see that the map $\psi_2 : \mathcal{W}_2 \rightarrow \mathcal{V}_2$ is given by the equations

$$\begin{aligned} (1 + \lambda Y_1)^p &= 1 + \lambda^p X_1 \\ (F_1(Y_1) + \lambda Y_2)^p (1 + \lambda Y_1)^{-1} &= G_1(\psi_1(Y_1)) + \lambda^p X_2. \end{aligned}$$

Note that ψ_1 is the same as ψ from (6.1), that is, $\psi_1(Y_1) = \frac{(1 + \lambda Y_1)^p - 1}{\lambda^p}$. Let $\alpha : \text{Spec } B \rightarrow \mathcal{V}_2$ be given by $X_1 \rightarrow T^{-u_1}$ and $X_2 \rightarrow 0$. Then, in

Proposition 6.15, we have

$$(6.7) \quad C \cong B[Y_1, Y_2]/(\psi_1(Y_1) - T^{-u_1}, (F_1(Y_1) + \lambda Y_2)^p - (1 + \lambda Y_1)G_1(T^{-u_1})),$$

and $\text{Spec } C \rightarrow \text{Spec } B$ is an unramified cover, which must reduce to some \mathbb{Z}/p^2 -cover of $k((t))$ if t is the reduction of T .

An intricate calculation ([GM98, Lemmas 5.2, 5.3]) shows that (6.7) in fact reduces to the \mathbb{Z}/p^2 -extension $A/k((t))$ in Proposition 6.5 with $x_1 = t^{-u_1}$ and $x_2 = 0$, which has upper jumps (u_1, pu_1) . By Lemma 4.3, the degree of the different of this extension is

$$(p^2 - 1)(u_1 + 1) + p(p - 1)^2 u_1.$$

Now, if we let $Z_1 = 1 + \lambda Y_1$ and $Z_2 = F_1(Y_1) + \lambda Y_2$, then we can write the equations for C as

$$(6.8) \quad Z_1^p = 1 + \lambda^p T^{-u_1}, \quad Z_2^p = Z_1 \cdot G_1(T^{-u_1}).$$

Letting C' be the normalization of $R[[T]]$ in $\text{Frac}(C)$, we see that $\text{Spec}(C' \otimes_R K) \rightarrow \text{Spec}(R[[T]] \otimes_R K)$ is branched of order p^2 at $u_1 + 1$ points (the zeroes and poles of $1 + \lambda^p T^{-u_1}$ in the open unit disc around $T = 0$) and of order p at $(p - 1)u_1$ points (the zeroes of $G_1(T^{-u_1})$, or $\exp_p(\mu^p T^{-u_1})$, in the open unit disc around $T = 0$). Since $\text{char}(K) = 0$, the degree of the different of this extension is also

$$(p^2 - 1)(u_1 + 1) + p(p - 1)^2 u_1.$$

By the different criterion (Proposition 4.8), $C'/R[[T]]$ is a lift of $A'/k[[t]]$, where A' is the integral closure of $k[[t]]$ in A .

Green and Matignon then show ([GM98, Lemma 5.4, Theorem 5.5]) that C can be deformed so as to find a lift of the integral closure of $k[[t]]$ in the \mathbb{Z}/p^2 -extension $A/k[[t]]$, corresponding to $x_1 = t^{-u_1}$ and x_2 arbitrary in Proposition 6.5. This is a subtle calculation, and we remark that the naïve approach (i.e., choosing $\alpha : \text{Spec } B \rightarrow \mathcal{V}_2$ to send X_2 to a lift of x_2 instead of to 0) fails because the different criterion is no longer satisfied in general. We will discuss this further in §6.6.

6.6. A different approach to lifting \mathbb{Z}/p^n -extensions

6.6.1. *The general form.* Since the explicit polynomials F_n and G_n involved in Sekiguchi-Suwa theory (Proposition 6.12) get very complicated when n gets large, it is difficult to generalize the methods of Green and Matignon to show that \mathbb{Z}/p^n is a local Oort group. In (as of yet unpublished) work of Stefan Wewers and the author, a different, less

explicit approach is taken. The starting point is equation (6.8). Generalizing this, it is clear that if R contains the p^n th roots of unity, then any \mathbb{Z}/p^n -extension of $\text{Frac}(R[[T]])$ can be given by equations

$$(6.9) \quad \begin{aligned} Z_1^p &= H_1(T) \\ Z_2^p &= Z_1 H_2(T) \\ &\vdots \\ Z_n^p &= Z_{n-1} H_n(T), \end{aligned}$$

where $H_i(T) \in \text{Frac}(R[[T]])$ for $1 \leq i \leq n$. The Galois action of a generator of \mathbb{Z}/p^n sends Z_i to $\zeta_{p^i} Z_i$, where ζ_{p^i} is a p^i th root of unity and $\zeta_{p^j}^{p^{j-i}} = \zeta_{p^i}$ for $j \geq i$. This is isomorphic to the extension $Z^{p^n} = H_1(T) H_2(T)^p \cdots H_n(T)^{p^{n-1}}$ in standard Kummer form.

The next proposition gives a somewhat explicit criterion for when a birational lift of a cyclic extension is an actual lift.

Proposition 6.17. (i) *Let $A/k[[t]]$ be a \mathbb{Z}/p^n -extension with upper jumps (u_1, \dots, u_n) . Suppose $A_R/R[[T]]$ is a birational lift of $A/k[[t]]$ given by normalizing $R[[T]]$ in the extension of fraction fields given by (6.9). If the H_i are polynomials in T^{-1} , if $\deg H_1 = u_1$, and if $\deg H_i = u_i - u_{i-1}$ for $i > 1$, then $A_R/R[[T]]$ is a lift of $A/k[[t]]$.*

(ii) *Suppose that $A_R/R[[T]]$ is the normalization of $R[[T]]$ in the extension of fraction fields generated by (6.9), where the H_i are polynomials in T^{-1} with $\deg H_1 = u_1$, $\deg H_i = u_i - u_{i-1}$ for $2 \leq i \leq n-1$, and $\deg H_n = (p-1)u_{n-1}$. Suppose further that the \mathbb{Z}/p^{n-1} -subextension $B_R/R[[T]]$ of $A_R/R[[T]]$ reduces to a \mathbb{Z}/p^{n-1} -extension $B/k[[t]]$ with upper jumps (u_1, \dots, u_{n-1}) . If the reduction $A/k[[t]]$ of $A_R/R[[T]]$ to characteristic p gives a separable extension of fraction fields, then A is integrally closed and the upper jumps of $A/k[[t]]$ are $(u_1, \dots, u_{n-1}, pu_{n-1})$. Thus $A_R/R[[T]]$ is a lift of some \mathbb{Z}/p^n -extension $A/k[[t]]$.*

Proof. To (i): If $T = x$ is in the open unit disc \mathcal{D} , and i is minimal such that $T = x$ is a zero of $H_i(T^{-1})$, then the branching index of $T = x$ in $\text{Spec}(A_R \otimes_R K) \rightarrow \text{Spec}(R[[T]] \otimes_R K)$ is at most p^{n-i+1} . Clearly $T = 0$ is the only pole of any $H_i(T^{-1})$, and has branching index at most p^n . Thus, the zeroes of $H_1(T^{-1})$ in \mathcal{D} , as well as $T = 0$, are branched of index at most p^n . The zeroes of $H_2(T^{-1})$ that are not zeroes of $H_1(T^{-1})$ are branched of index at most p^{n-1} , etc. Since H_i has at most $\deg H_i$ zeroes, it follows that the degree δ_η of the different of $(A_R \otimes_R K)/(R[[T]] \otimes_R K)$ satisfies

$$(6.10) \quad \delta_\eta \leq (p^n - 1)(u_1 + 1) + \sum_{i=2}^n (p^{n-i+1} - 1)p^{i-1}(u_i - u_{i-1}).$$

By Lemma 4.3, the right-hand side is the degree δ_s of the different of $A/k[[t]]$. By the different criterion (Proposition 4.8), we have $\delta_\eta = \delta_s$, and our birational lift is an actual lift.

To (ii): As in part (i), the degree δ_η of the different of $(A_R \otimes_R K)/(R[[T]] \otimes_R K)$ satisfies
(6.11)

$$\delta_\eta \leq (p^n - 1)(u_1 + 1) + \left(\sum_{i=2}^{n-1} (p^{n-i+1} - 1)p^{i-1}(u_i - u_{i-1}) \right) + (p-1)p^{n-1}((p-1)u_{n-1}).$$

Let (u_1, \dots, u_n) be the upper jumps of $A'/k[[t]]$, where A' is the integral closure of A . Then the degree of the different of $A'/k[[t]]$ is

$$\delta'_s := (p^n - 1)(u_1 + 1) + \sum_{i=2}^n (p^{n-i+1} - 1)p^{i-1}(u_i - u_{i-1}),$$

by Lemma 4.3. If δ_s is the degree of the different of $A/k[[t]]$, then $\delta'_s \leq \delta_s = \delta_\eta$. But, by Corollary 6.6, we have $\delta'_s \geq \delta_\eta$, with equality iff $u_n = pu_{n-1}$. So $\delta_s = \delta'_s$, which means that $A = A'$ and the u_i are as desired. \square

Remark 6.18. Since $\delta_\eta = \delta_s$ in both cases above, we have equalities in (6.10) and (6.11), so each of the zeroes of each $H_i(T^{-1})$ must be simple and must lie in the open unit disc around $T = 0$. This means that, up to scaling by a constant, each polynomial $H_i(T^{-1})$ is of the form $a_0 + a_1 T^{-1} + \dots + a_{N_i} T^{-N_i}$ with $v(a_0) = 0$ and $v(a_j) > 0$ for $j > 0$. Furthermore, the zeroes of all of the $H_i(T^{-1})$ must be pairwise distinct.

Note that Proposition 6.17(ii) applies to Green and Matignon's lift in (6.8), as $G_1(\psi_1(Y_1)) = \exp_p(\mu^p T^{-u_1})$. In fact, Proposition 6.17(i) applies to Green and Matignon's lifts for general \mathbb{Z}/p^2 -extensions (this is how they prove that the birational formulas they write down are actually lifts). So it seems reasonable, given an \mathbb{Z}/p^n -extension, to look for a lift in the form of (6.9) satisfying the conditions of Proposition 6.17(i).

Furthermore, Proposition 6.17 suggests a framework to attack the lifting problem for cyclic extensions:

Framework 6.19. Suppose $A/k[[t]]$ is a \mathbb{Z}/p^n -extension with upper jumps (u_1, \dots, u_n) . Let $B/k[[t]]$ be the \mathbb{Z}/p^{n-1} -subextension. Assume by induction that we have a lift $B_R/R[[T]]$ of $B/k[[t]]$ given by equations in the form of (6.9), where the H_i are polynomials in T^{-1} with $\deg H_1 = u_1$, and $\deg H_i = u_i - u_{i-1}$ for $2 \leq i \leq n-1$.

Step 1: We seek a polynomial H_n in T^{-1} of degree $(p-1)u_{n-1}$ so that the reduction of (6.9) is separable. By Proposition 6.17(ii), the equations (6.9) then lift *some* \mathbb{Z}/p^n -extension with upper jumps $(u_1, \dots, u_{n-1}, pu_{n-1})$ whose \mathbb{Z}/p^{n-1} -subextension is $B/k[[t]]$.

Step 2: If we can find such an H_n , then the goal is to replace H_n by a polynomial of degree $u_n - u_{n-1}$ in order to give a birational lift of $A/k[[t]]$, using an argument along the lines of that of Green and Matignon. This will be an actual lift by Proposition 6.17.

6.6.2. Depth and separability. Framework 6.19 forces us to understand when an extension of $R[[T]]$ has separable reduction. In Appendix B, the notion of the *depth* of a cyclic \mathbb{Z}/p^n -extension $A_R/R[[T]]$ is discussed. This is a non-negative number that is 0 iff the extension has separable reduction.

Fix a \mathbb{Z}/p^n -extension $A_R/R[[T]]$, corresponding to a morphism $f_R : \text{Spec } A_R \rightarrow \text{Spec } R[[T]]$, birationally given by equations in the form (6.9). For each i , $1 \leq i \leq n$, let $(A_R)_i/R[[T]]$ be the unique \mathbb{Z}/p^i -subextension, and let $(f_R)_i$ be the corresponding morphism. For each rational $r \geq 0$, let a_r be an element of a finite extension of R of valuation r , and enlarge R so that it contains a_r . Then $\mathcal{D}_r := \text{Spec } R[[a_r^{-1}T]]$ is the open disc of radius $|a_r|$ (§A.1). Note that $\mathcal{D} = \mathcal{D}_0$. The restriction of any $(f_R)_i$ to the disc \mathcal{D}_r corresponds to making a substitution $U = a_r^{-1}T$ in the $H_1(T^{-1}), \dots, H_i(T^{-1})$, and viewing the equations (6.9) as giving an extension of $R[[U]]$. We denote the depth of such an extension by $\delta_i(r)$, and we view δ_i as a function of r . Our goal is to find H_1, \dots, H_n such that $\delta_n(0) = 0$.

The following result is unpublished, but is known to the experts. It can essentially be derived from [Obu09a, §5.3], where the depth is called the “effective different.”

Lemma 6.20. *Let $(f_R)_i$ and a_r be as above. For each $1 \leq i \leq n$, the depth δ_i is a piecewise-linear function of r . The right-derivative of δ_i at r is less than or equal to $\nu_i(r) - 1$, where $\nu_i(r)$ is the number of branch points of the generic fiber of $(f_R)_i$ with valuation greater than r (in terms of the coordinate T). Equality holds iff the special fiber of $(f_R)_i|_{\mathcal{D}_r}$ is smooth (i.e., the integral closure of $R[[a_r^{-1}T]]$ in $\text{Frac}((A_R)_i)$ has integrally closed reduction).*

Let us now place ourselves in Framework 6.19 and use the notation therein. We will make the further assumption (for the remainder of §6.6.2) that the zeroes of H_1, \dots, H_{n-1} (and thus the branch points of the generic fiber of $(f_R)_{n-1}$) all have valuation greater than $\frac{1}{u_{n-1}(p-1)}$. This assumption is essential (in fact, its necessity is the reason that we do not have a full proof that \mathbb{Z}/p^n is a local Oort group), as it has the consequence that $\delta_{n-1}(\frac{1}{u_{n-1}(p-1)}) = \frac{1}{p-1}$. This follows from Lemma 6.20. Then, [Wew11, Proposition 6.1(iii)] shows that, no matter what we pick for H_n , we will have $\delta_n(\frac{1}{u_{n-1}(p-1)}) = \frac{p}{p-1}$ (here δ_{n-1} and δ_n play the roles of $\bar{\delta}$ and δ in [Wew11]).

Lemma 6.21. *If we pick H_n of degree $(p-1)u_{n-1}$ in T^{-1} (as in Framework 6.19) so that all of its zeroes are simple and have valuation $\frac{1}{u_{n-1}(p-1)}$, then $\delta_n(0) = 0$ iff the special fiber of f_R is smooth.*

Proof. By Remark 6.18, we see that f_R has a total of $pu_{n-1} + 1$ branch points on the generic fiber (corresponding to the zeroes of $H_1(T^{-1}), \dots, H_n(T^{-1})$, which are all simple and distinct, and the unique shared pole $T = 0$). Since $\delta_n(\frac{1}{u_{n-1}(p-1)}) = \frac{p}{p-1}$, Lemma 6.20 shows that $\delta_n(0) = 0$ iff the special fiber of f_R is smooth. \square

The next proposition is a major result in [OW11].

Proposition 6.22. *Given H_n as in Lemma 6.21, we can adjust H_n , while keeping its degree and the valuation of its zeroes the same, so that the special fiber of f_R becomes smooth.*

Very rough sketch of proof. If f_R is not smooth, there is a minimum $r \geq 0$ corresponding to a maximal disk \mathcal{D}_r such that $(f_R)|_{\mathcal{D}_r}$ is smooth. One can show using [Wew11, Proposition 6.1] that $r \leq \frac{1}{u_{n-1}(p-1)}$. One then shows that if $r > 0$ and R is taken large enough, then one can adjust H_n as in the proposition to decrease r (we think of r as depending on H_n). This involves solving differential equations over k , which in turn reduces to solving systems of linear equations over \mathbb{F}_p . Amazingly, these systems are overdetermined yet still always have solutions! Lastly, one independently shows that there is some particular H_n as in the proposition where r takes a minimum (and H_n can be realized over some finite extension $R/W(k)$). This minimum must be 0, and this is the H_n we seek. \square

Remark 6.23. Note that this proof is non-constructive. In particular, we have no control over how large we must take R .

Corollary 6.24. *Suppose $B/k[[t]]$ is a \mathbb{Z}/p^{n-1} -extension with upper jumps (u_1, \dots, u_{n-1}) that lifts to characteristic zero, and that the lift to*

characteristic zero is in the form of (6.9) with H_1, \dots, H_{n-1} polynomials in T^{-1} with $\deg H_1 = u_1$ and $\deg H_i = u_i - u_{i-1}$ for $2 \leq i \leq n-1$. Suppose further that all the zeroes of $H_1(T^{-1}), \dots, H_{n-1}(T^{-1})$ have valuation greater than $\frac{1}{u_{n-1}(p-1)}$. Then there is a \mathbb{Z}/p^n -extension $A_0/k[[t]]$ with upper jumps $(u_1, \dots, u_{n-1}, pu_{n-1})$ and a \mathbb{Z}/p^{n-1} -subextension $B/k[[t]]$ that lifts to characteristic zero.

Proof. Follows from Step 1 of Framework 6.19, Lemma 6.21, and Proposition 6.22. \square

6.6.3. Lifting general extensions. By Corollary 6.24, we can, under mild hypotheses, extend a lift of a \mathbb{Z}/p^{n-1} -extension $B/k[[t]]$ to a lift of some \mathbb{Z}/p^n -extension $A_0/k[[t]]$ with minimal possible n th upper jump. The extension $A_0/k[[t]]$ can be written in the form of Proposition 6.5, where $\deg x_n < pu_{n-1}$. Its lift can be written in the form of (6.9) with H_n a polynomial of degree $(p-1)u_{n-1}$ in T^{-1} . Our goal is now to show that we can, in fact, extend our lift of $B/k[[t]]$ to lift *any* \mathbb{Z}/p^n -extension, $A/k[[t]]$ (with $B/k[[t]]$ as the \mathbb{Z}/p^{n-1} -subextension) no matter what the upper jump. In particular, if the n th upper jump of $A/k[[t]]$ is u_n , we wish to replace H_n with a polynomial of degree $u_n - u_{n-1}$ in T^{-1} that yields a birational lift of $A/k[[t]]$. By Proposition 6.17 (i), this gives an actual lift of $A/k[[t]]$.

Let us recall that our lift of the \mathbb{Z}/p -extension given birationally by $y^p - y = t^{-u_1}$ was given by the equation $Z^p = 1 + \lambda^p T^{-u_1}$ (§6.3). Now, suppose we were to replace t^{-u_1} by $t^{-u_1} + q(t^{-1})$, where q has no terms of degree divisible by p . If $Q(T^{-1})$ is any lift of $q(t^{-1})$ to $R[T^{-1}]$ that preserves the degree, the argument of Theorem 6.8 easily carries through to show that $Z^p = 1 + \lambda^p(T^{-u_1} + Q(T^{-1}))$ gives a lift of the extension given by $y^p - y = t^{-u_1} + q(t^{-1})$.

The general situation is similar. In fact, in [OW11], we prove the following result:

Proposition 6.25. *Suppose $A/k[[t]]$ is obtained from $A_0/k[[t]]$ as above by replacing x_n in Proposition 6.5 with $x_n + q(t^{-1})$, where q has no terms of degree divisible by p . If $Q(T^{-1})$ is any lift of $q(t^{-1})$ to $R[T^{-1}]$, then replacing H_n by $H_n + \lambda^p Q(T^{-1})$ in (6.9) gives a birational lift of $A/k[[t]]$.*

Unfortunately, we cannot apply Proposition 6.17(i) in general. For instance, if $\deg q > pu_{n-1}$, then $\deg q = u_n$, thus $\deg Q \geq u_n$ and $\deg(H_n + \lambda^p Q) \geq u_n$. But to apply Proposition 6.17(i), we would need $\deg(H_n + \lambda^p Q) = u_n - u_{n-1}$. However, it turns out that there is some polynomial H'_n of degree $u_n - u_{n-1}$ in T^{-1} , as well as some lift Q of q to $R[[T]]$, such that the ratio of H'_n to $H_n + \lambda^p Q$ is a p th power

in $\text{Frac}(R[[T]])$. Replacing H_n by H'_n must give the same extension of $\text{Frac}(R[[T]])$ as replacing H_n by $H_n + \lambda^p Q$. By Proposition 6.25, replacing H_n by H'_n gives a birational lift, and by Proposition 6.17(i), this is an actual lift. Thus we have completed step 2 of Framework 6.19. Explicitly:

Proposition 6.26. *Suppose $B/k[[t]]$ is a \mathbb{Z}/p^{n-1} -extension with upper jumps (u_1, \dots, u_{n-1}) that lifts to characteristic zero, and that the lift to characteristic zero is in the form of (6.9) with H_1, \dots, H_{n-1} satisfying the assumptions of Corollary 6.24. Then any \mathbb{Z}/p^n -extension $A/k[[t]]$ with \mathbb{Z}/p^{n-1} -subextension $B/k[[t]]$ lifts to characteristic zero.*

However, in order to continue the induction, it is necessary that the zeroes of $H'_n(T^{-1})$ have valuation greater than $\frac{1}{u_n(p-1)}$. In our construction of H'_n , this is regrettably not always the case. The best we can do so far is the following:

Lemma 6.27. *Suppose that there is no $a \in p\mathbb{Z}$ such that $u_n - pu_{n-1} < a \leq (u_n - pu_{n-1}) \left(\frac{u_n}{u_n - u_{n-1}} \right)$. Then the lift of $A/k[[t]]$ in Proposition 6.26 can be accomplished in the form of (6.9) with H_n a polynomial in T^{-1} of degree $u_n - u_{n-1}$ and the zeroes of $H_n(T^{-1})$ having valuation greater than $\frac{1}{u_n(p-1)}$.*

Using Lemma 6.27, we can state our strongest result about the local lifting problem for \mathbb{Z}/p^n .

Theorem 6.28 ([OW11]). *Let $A/k[[t]]$ be a \mathbb{Z}/p^n -extension with upper jumps (u_1, \dots, u_n) . Then $A/k[[t]]$ lifts to characteristic zero so long as, for each i , $3 \leq i \leq n-1$, there does not exist $a_i \in p\mathbb{Z}$ with*

$$u_i - pu_{i-1} < a_i \leq (u_i - pu_{i-1}) \left(\frac{u_i}{u_i - u_{i-1}} \right).$$

Proof. The cases $n = 1$ and $n = 2$ are given by Theorems 6.8 and 6.16. Furthermore, the lifts given explicitly in Theorem 6.8 and by Green and Matignon for Theorem 6.16 are in the form of (6.9) where $\deg H_1 = u_1$, $\deg H_2 = u_2 - u_1$, and the zeroes of $H_1(T^{-1})$ and $H_2(T^{-1})$ have valuation greater than $\frac{1}{u_1(p-1)}$ and $\frac{1}{u_2(p-1)}$, respectively. The proof then follows by repeatedly applying Proposition 6.26 and Lemma 6.27 \square

Remark 6.29. (i) The condition in Theorem 6.28 is vacuous for $n = 3$, so \mathbb{Z}/p^3 is a local Oort group.

(ii) The condition is satisfied if $u_i = pu_{i-1}$ for all $i \geq 3$. In particular, it holds for all “minimal” sets of upper jumps $(u_1, pu_1, \dots, p^{n-1}u_1)$.

- (iii) The example $p = 5$, $n = 4$, $(u_1, \dots, u_4) = (1, 5, 34, 170)$, and $a_3 = 10$ shows that the condition is not always satisfied.
- (iv) The condition is somewhat strange: for a \mathbb{Z}/p^4 -extension, the only upper jumps that need to be checked are the second and third! Because of this strangeness, it seems reasonable to believe that the condition is not necessary, and that the Oort conjecture still holds.
- (v) Our proof gives no insight into the ring R necessary for lifting to characteristic zero. To show that a lift is possible over $R = W(k)[\zeta_{p^n}]$ will require new techniques.

§7. Metacyclic groups

Throughout §7, $G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$ with $p \nmid m$ and G not cyclic. Also, $R/W(k)$ is a large enough finite extension and $K = \text{Frac}(R)$. Let $\mathcal{D} = \text{Spec } R[[T]]$ be the open unit disc. Bouw, Wewers, and Zapponi have proven the following theorem (although they did not phrase it in terms of the KGB obstruction).

Theorem 7.1 ([BWZ09], Theorem 2.1). *A G -extension $A/k[[t]]$ lifts to characteristic zero iff its KGB obstruction vanishes. In other words, $A/k[[t]]$ lifts exactly when the (unique) positive jump in the higher ramification filtration for the lower numbering of $A/k[[t]]$ is congruent to $-1 \pmod{m}$.*

The following corollary immediately follows from Theorems 5.2 and 7.1.

Corollary 7.2. *For odd primes p , the dihedral group D_p is a local Oort group.*

Note that, as a consequence of Proposition 5.9, the KGB obstruction can only vanish if the conjugation action of \mathbb{Z}/m on \mathbb{Z}/p is faithful. In particular, we must have $m \mid (p-1)$. In light of this, we view the m th roots of unity as living in $(\mathbb{Z}/p)^\times \subseteq k^\times$.

7.1. An explicit lifting example

We present an example of a G -extension that lifts to characteristic zero, due to Green and Matignon.

Proposition 7.3 ([GM99], IV Prop. 2.2.1). *Let p be an odd prime. The D_p -extension of $k[[t]]$ given birationally by the equations*

$$(7.1) \quad x^2 = t, \quad y^p - y = x^{-1},$$

lifts to characteristic zero (in fact, one can take $R = W(k)[\zeta]$, where ζ is a primitive p th root of unity).

Proof. There are elements $\sigma, c \in D_p$ with

$$\sigma(x) = x, \sigma(y) = y + 1, c(x) = -x, c(y) = -y.$$

Let ζ_p be a primitive p th root of unity, and let $\lambda = \zeta_p - 1$. We claim the extension of $R[[T]]$ given birationally by equations

$$(7.2) \quad \left(X + \frac{\lambda^p}{2}\right)^2 = T, \quad Z^p = 1 + \lambda^p X^{-1}$$

gives a lift, under the automorphisms

$$\sigma(X) = X, \sigma(Z) = \zeta_p Z, c(X) = -X - \lambda^p, c(Z) = \frac{1}{Z}.$$

A straightforward calculation verifies that the given actions of σ and c indeed preserve the equations (7.2). It is also clear that the actions of σ and c generate a dihedral group. Upon making the substitution $1 + \lambda Y = Z$, we see that the equations (7.2) reduce to the equations (7.1), as in the proof of Theorem 6.8. Since $c(Y) = \frac{-Y}{1 + \lambda Y}$ and $\sigma(Y) = \zeta_p Y + 1$, we see that the action of D_p in characteristic zero reduces to the action of D_p in characteristic p . Lastly, one checks that the unique positive lower jump on the special fiber is 1 (Proposition 6.5), so the degree of the different in characteristic p is $3p - 2$ by (4.1). The generic fiber is branched at $T = (\frac{\lambda^p}{2})^2$ of index p and at $T = 0$ of index 2, so the degree of the different is also $3p - 2$. By the different criterion (Proposition 4.8), the equations (7.2) give a lift. \square

Remark 7.4. Similar examples are given in [GM99, Proposition 2.2.2] of $\mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$ -extensions that lift, for any p . In general, however, it is not known how to find explicit equations for lifting G -extensions, where $G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$.

7.2. Structure theorems

Our first step toward the proof of Theorem 7.1 is a structure result completely classifying G -extensions of $k((t))$.

Lemma 7.5 ([Pri03], Lemma 2.1.2, [Pri02], Lemma 1.4.1 (iv)). *Suppose $L/k((t))$ is a $G = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ -extension with unique positive lower jump h . Then, up to a change in the parameter t , we have*

$$L \cong k((t))[u, y]/(u^m - t, y^p - y - u^{-h}).$$

Furthermore, given a primitive m th root of unity ζ_m (viewed as an element of $(\mathbb{Z}/p)^\times$), we can choose elements $\sigma \in G$ of order p and $c \in G$ of order m such that the Galois action is given by

$$\sigma(u) = u, \sigma(y) = y + 1, c(u) = \zeta_m u, c(y) = \zeta_m^{-h} y.$$

Also, we have $c\sigma c^{-1} = \sigma^{\zeta_m^h}$.

In particular, up to isomorphism, there is only one G -extension of $k((t))$ with given positive lower jump.

A more general structure theorem, this one in characteristic zero, will be useful for the proof of Theorem 7.1. In particular, we can classify certain G -extensions $B/R[[T]]\{T^{-1}\}$. Note that $R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring with residue field $k((t))$, and if π is a uniformizer of R , then π is also a uniformizer of $R[[T]]\{T^{-1}\}$. As we will see, we only care about such extensions for which B is abstractly isomorphic to $R[[U]]\{U^{-1}\}$ (these correspond to extensions in which R is algebraically closed). Fix such an isomorphism.

We need three invariants to classify such extensions. The first is the *different* of the extension. For the second, let $\sigma \in G$ be an order p element. Then $\frac{\sigma(U)}{U} - 1$ can be written as $\pi^n f(U)$, where $f(U)$ has coefficients of nonnegative valuation and at least one coefficient of valuation zero. The *conductor* of $B/R[[T]]\{T^{-1}\}$ is the valuation of the reduction of $f(U)$ to $k((u))$ (where u has valuation 1). Lastly, if $c \in G$ generates a subgroup C of order m , then $c(T) \equiv \alpha T \pmod{T^2}$, where α reduces to a nonzero element $\bar{\alpha} \in k^\times$. This gives a character $C \rightarrow k^\times$, which is called the *tame inertia character*.

Lemma 7.6 ([BoW06], §2, in particular Proposition 2.3). *The different, conductor, and tame inertia character of G -extension of $R[[T]]\{T^{-1}\}$ are well-defined. Up to a change in the parameter T , there is at most one G -extension $R[[U]]\{U^{-1}\}/R[[T]]\{T^{-1}\}$ with a given different, conductor, and tame inertia character. This works even if G is cyclic.*

A G -extension of $R[[T]]\{T^{-1}\}$ with degree of different δ , conductor h , and tame inertia character λ is said to be *of type* (δ, h, λ) .

Remark 7.7. Unfortunately, Lemmas 7.5 and 7.6 do *not* hold if $\mathbb{Z}/p \rtimes \mathbb{Z}/m$ is replaced by $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, for $n > 1$. See Remark 7.14.

7.2.1. Outline of proof. We outline the proof of Theorem 7.1, which will depend on results from §7.3.

Proof of Theorem 7.1. Recall from §4.2 that if $A_R/R[[T]]$ is a lift of the G -extension $A/k[[t]]$, then A_R is abstractly isomorphic to $R[[U]]$, the ring of functions on an open unit disc. Furthermore, for any faithful G -action on $R[[U]]$ by continuous R -linear automorphisms (that is, a G -action on the open unit disc), the fixed ring $R[[U]]^G$ is abstractly isomorphic to $R[[T]]$. In Proposition 7.9 (using Definition 7.10), we show that a faithful G -action on the open unit disc $\text{Spec } R[[U]]$ with separable

reduction and positive conductor h at the boundary gives rise to a so-called *Hurwitz tree with conductor h* . In Proposition 7.11, we reverse this process, showing how a Hurwitz tree with conductor h gives rise to a G -action on the open unit disc. By Corollary 7.12, the G -extension $R[[U]]/R[[T]]$ associated to such a G -action reduces to a G -extension $A_h/k[[t]]$ with unique positive lower jump h .

In Proposition 7.13, we construct, for each $h \equiv -1 \pmod{m}$, a Hurwitz tree of conductor h , thus yielding a lift of a G -extension $A_h/k[[t]]$ with unique positive lower jump h . Note that any G -extension $A/k[[t]]$ with positive lower jump h is isomorphic to $A_h/k[[t]]$ (Lemma 7.5). Since we have constructed a lift of $A_h/k[[t]]$, there must also be a lift of $A/k[[t]]$. This completes the proof of Theorem 7.1. \square

7.3. Hurwitz trees

Suppose G acts faithfully on the open unit disc $\mathcal{D} = \operatorname{Spec} R[[U]]$ by R -automorphisms with no inertia above a uniformizer of R (that is, the action reduces to a faithful G -action on $\operatorname{Spec} k[[u]]$). The idea of a Hurwitz tree is to break the disc \mathcal{D} up into smaller discs, punctured discs, and annuli upon which G acts, to isolate the salient features of these actions in combinatorial form, and to keep track of how they connect with each other. Henrio gave the first major exposition of Hurwitz trees ([Hen99]), dealing with the case of a \mathbb{Z}/p -action on an open disc. The concept was extended by Bouw and Wewers to encompass G -actions. For the basic facts about discs and annuli that we will use, see Appendix A.1. Our concept of Hurwitz tree is more restrictive than that of [BoW06], in that what we call a Hurwitz tree is called a “Hurwitz tree of different 0” in [BoW06].

Throughout §7.3, let \mathcal{D}_K be the generic fiber of \mathcal{D} . Clearly, G acts on \mathcal{D}_K . Let σ be an element of G of order p .

7.3.1. The Hurwitz tree associated to a G -action on the open unit disc. Let $\partial\mathcal{D} = \operatorname{Spec} R[[U]]\{U^{-1}\}$ be the boundary of \mathcal{D} . Then a G -action on \mathcal{D} gives a G -action on $\partial\mathcal{D}$, with some conductor h . Then [GM98, I, Claim 3.3] shows that the number of (geometric) fixed points of the action of σ on \mathcal{D}_K is $h + 1$. We may assume that R (and K) are large enough so that the fixed points y_1, \dots, y_{h+1} of σ are all defined over K . We assume further that $h > 0$, so there are at least two fixed points. Lastly, we assume that G acts without inertia above a uniformizer of R . Now, let Y^{st} be the stable model of \mathbb{P}_K^1 corresponding to the marked disc $(\mathcal{D}; y_1, \dots, y_{h+1})$ (see §A.3—this is essentially the minimal stable model of \mathbb{P}_K^1 that separates the specializations of the y_i and ∞ , where $\mathcal{D}_K \subseteq \mathbb{P}_K^1$ is viewed as being the open unit disc centered at 0). Let \bar{Y} be

the special fiber of Y^{st} . If $\overline{\infty}$ is the specialization of the point $\infty \in \mathbb{P}_K^1$ to \overline{Y} , then let $\overline{V} = \overline{Y} \setminus \{\overline{\infty}\}$. The set of points of \mathbb{P}_K^1 that specialize to \overline{V} is a closed disc \mathcal{E} , strictly contained in \mathcal{D} . In fact, \mathcal{E} is the smallest closed disc containing all the points y_1, \dots, y_{h+1} .

Let \mathcal{E}^{st} be the model of \mathcal{E} whose special fiber is \overline{V} . Since the points y_1, \dots, y_{h+1} are permuted by G , we have that G acts on \mathcal{E}^{st} . Furthermore, since $\sigma \in G$ fixes each y_i , it follows from the stability of \overline{Y} that σ must fix the special fiber \overline{V} of \mathcal{E}^{st} pointwise.

Let $\mathcal{D}' = \mathcal{D}/\langle\sigma\rangle$, that is,

$$\mathcal{D}' = \operatorname{Spec} R[[U]]^{\langle\sigma\rangle} \cong \operatorname{Spec} R[[T]]$$

for some parameter T . Then \mathcal{D}' is an open unit disc with generic fiber \mathcal{D}'_K . Let z_1, \dots, z_{h+1} be the images of y_1, \dots, y_{h+1} under the canonical map $\mathcal{D} \rightarrow \mathcal{D}'$. As above, we let Z^{st} be the stable model of \mathbb{P}_K^1 corresponding to the marked disc $(\mathcal{D}'; z_1, \dots, z_{h+1})$, with special fiber \overline{Z} . If $\overline{\infty}'$ is the specialization of ∞ to \overline{Z} and $\overline{W} = \overline{Z} \setminus \{\overline{\infty}'\}$, then we define \mathcal{E}' and $(\mathcal{E}')^{st}$ in analogy to \mathcal{E} , \mathcal{E}^{st} in the previous paragraph. We have $\mathcal{E}' = \mathcal{E}/\langle\sigma\rangle$ and $(\mathcal{E}')^{st} = \mathcal{E}^{st}/\langle\sigma\rangle$. Note that the map $\mathcal{E}^{st} \rightarrow (\mathcal{E}')^{st}$ reduces to a purely inseparable map $\overline{V} \rightarrow \overline{W}$ on the special fiber, as σ fixes \overline{V} pointwise.

We build the *Hurwitz tree* corresponding to the G -action on \mathcal{D} as follows (note that we will not explicitly define a Hurwitz tree until the next section). Consider the *dual graph* Γ of \overline{Z} , whose edges $E(\Gamma)$ and vertices $V(\Gamma)$ correspond to the irreducible components and nodes of \overline{Z} , respectively. An edge connects two vertices if the corresponding node is the intersection of the two corresponding components. We append another vertex and edge v_0 and e_0 so that e_0 connects v_0 to the vertex corresponding to the component containing $\overline{\infty}'$.

If $e \neq e_0$ is in $E(\Gamma)$, then ϵ_e is defined to be the thickness of the corresponding node of \overline{Y} (§A.1). We set ϵ_{e_0} to be the thickness of the open annulus $\mathcal{D} \setminus \mathcal{E}$. If $v \neq v_0$ is in $V(\Gamma)$, then let η' be the generic point of the component \overline{S}_v of \overline{Z} corresponding to v , and let η be the point of \overline{Y} lying above η' . Then the extension $\hat{\mathcal{O}}_{\overline{Y}^{st}, \eta} / \hat{\mathcal{O}}_{\overline{Z}^{st}, \eta'}$ gives an inseparable extension on residue fields and satisfies the conditions of Proposition B.1. We set δ_v to be the valuation of the different of this extension (Proposition B.1) and ω_v to be the corresponding deformation datum (§B.2). We have that ω_v is a meromorphic differential form on \overline{S}_v . We set $\delta_{v_0} = 0$, and we do not define ω_{v_0} .

Lastly, we note that if we choose a subgroup $C \cong \mathbb{Z}/m \subseteq G$, then C acts on \mathcal{D}' , \mathcal{E}' , and $(\mathcal{E}')^{st}$. So C acts on \overline{Z} , as well as on Γ . Also, if $c \in C$,

then $c^*(\omega_{c(v)})$ is a differential form on the component corresponding to v .

Definition 7.8. The data consisting of the group $C \subseteq G$, the curve \bar{Z} , the marking ∞' , the specializations $\bar{z}_1, \dots, \bar{z}_{h+1}$ of z_1, \dots, z_{h+1} to \bar{Z} , the graph Γ , the differential forms ω_v for $v \in V(\Gamma) \setminus \{v_0\}$, the numbers δ_v for $v \in V(\Gamma)$ and ϵ_e for $e \in E(\Gamma)$, and the C -action on \bar{Z} , comprise the *Hurwitz tree* of the G -action on \mathcal{D} .

7.3.2. Hurwitz trees in general. The data of Definition 7.8 satisfy many compatibilities. We will define a general Hurwitz tree to be a collection of data in the form above satisfying these compatibilities.

Proposition 7.9. *The data in the Hurwitz tree of Definition 7.8 satisfy the following properties:*

- (i) \bar{Z} has genus 0 and is stably marked by $\infty, \bar{z}_1, \dots, \bar{z}_{h+1}$ (i.e., each irreducible component contains at least three points that are either marked or singular).
- (ii) For each $v \in V(\Gamma)$, we have that δ_v is rational and satisfies $0 \leq \delta_v \leq 1$, with $\delta_v = 0$ iff $v = v_0$.
- (iii) For each $v \in V(\Gamma) \setminus \{v_0\}$, the divisor of ω_v is supported at the marked points and the singular points of \bar{Z} . In particular, ω_v has simple poles at any of the points $\bar{z}_1, \dots, \bar{z}_{n+1}$ on the component of \bar{Z} corresponding to v .
- (iv) If $\delta_v = 1$, then ω_v is logarithmic. Otherwise, ω_v is exact.
- (v) If $e \in E(\Gamma)$, then ϵ_e is a positive rational number.
- (vi) If a node \bar{z} of \bar{Z} lies on two components \bar{S}' and \bar{S}'' corresponding to vertices v' and v'' of Γ , then $\text{ord}_{\bar{z}}\omega_{v'} + \text{ord}_{\bar{z}}\omega_{v''} = -2$.
- (vii) In the situation of (vi), if $v' \neq v_0$ and the node \bar{z} corresponds to an edge $e \in E(\Gamma)$, then

$$\delta_{v'} - \delta_{v''} = (p-1)\epsilon_e(\text{ord}_{\bar{z}}\omega_{v'} + 1).$$

- (viii) The action of C fixes ∞' and permutes the points $\bar{z}_1, \dots, \bar{z}_{h+1}$.
- (ix) If $C_{\bar{z}} \subseteq C$ is the stabilizer of a node \bar{z} of \bar{Z} , then the characters describing the action of $C_{\bar{z}}$ on the tangent spaces of the two components of \bar{Z} intersecting at \bar{z} are inverse to each other.
- (x) If χ is the conjugation character of C on $\mathbb{Z}/p \subseteq G$ (§B.2), then $c^*\omega_{c(v)} = \chi(c)\omega_v$ for all c, v .

Proof. Parts (i), (iv), and (v) follow from the definitions. Part (ii) follows from Proposition B.1. Part (iii) follows from [Wew03, Prop. 1.7]. Parts (vi) and (vii) follow from [Hen99, Ch. 5]. Parts (viii) and (ix) follow because the Hurwitz tree comes from an action of G . For a proof of part (x), see [BoW06, Prop. 3.4]. \square

Definition 7.10. Suppose \bar{Z} is a semistable curve over k of genus 0 with smooth distinct marked points $\bar{\infty}', \bar{z}_1, \dots, \bar{z}_{h+1}$. Let Γ be the dual graph of \bar{Z} , with an extra vertex v_0 (called the *root vertex*) and an extra edge e_0 connecting v_0 to the vertex corresponding to the component containing $\bar{\infty}'$. Suppose that each vertex $v \in V(\Gamma)$ has an associated meromorphic differential form ω_v on the corresponding component (unless $v = v_0$) and an associated rational number δ_v . Suppose that each edge $e \in E(\Gamma)$ has an associated thickness ϵ_e . Lastly, suppose that the group $C \cong \mathbb{Z}/m$ acts on \bar{Z} (thus on Γ), and has an injective character $\chi : C \rightarrow \mathbb{F}_p^\times$ such that $c^* \omega_{c(v)} = \chi(c) \omega_v$. If this data satisfies parts (i)–(x) of Proposition 7.9, then it is called a *Hurwitz tree of type (C, χ)* . The *conductor* of this Hurwitz tree is defined to be h . We denote the Hurwitz tree by $(\bar{Z}; \bar{\infty}', \bar{z}_1, \dots, \bar{z}_{h+1}; \omega_v; \delta_v; \epsilon_e)$.

Note that, as was mentioned at the beginning of §7.3.1, the conductor of a G -action on the open unit disc at the boundary is equal to the conductor of its associated Hurwitz tree.

7.3.3. Lifting Hurwitz trees. The main result of this section is the following:

Proposition 7.11 ([BoW06], Theorem 3.6). *Suppose we are given a Hurwitz tree $\mathcal{T} = (\bar{Z}; \bar{\infty}', \bar{z}_1, \dots, \bar{z}_{h+1}; \omega_v; \delta_v; \epsilon_e)$ of type $(C \cong \mathbb{Z}/m, \chi)$ with conductor $h > 0$, and χ an injective character $C \rightarrow \mathbb{F}_p^\times$. If $G \cong \mathbb{Z}/p \rtimes_\chi C$, then there is a G -action on the open unit disc $\mathcal{D} = \text{Spec } R[[U]]$, with no inertia above a uniformizer of R , whose associated Hurwitz tree (Definition 7.8) is \mathcal{T} . In particular, the conductor of the action of G at the boundary $\partial \mathcal{D} = \text{Spec } R[[U]]\{U^{-1}\}$ is h .*

Very rough sketch of proof. The edges $e \in E(\Gamma)$ of the graph Γ of \mathcal{T} correspond to open annuli \mathcal{A}_e , whereas the vertices $v \in V(\Gamma)$ (other than the root vertex) correspond to (possibly punctured) closed discs \mathcal{U}_v , each \mathcal{U}_v being a closed disc with $r_v - 1$ open discs of the same radius removed, where r_v is the number of edges incident to v (see §A.2). These annuli and punctured discs fit together to form the open unit disc. Furthermore, for each \mathcal{U}_v , we can use δ_v and ω_v to construct a \mathbb{Z}/p -cover of \mathcal{U}_v . Because of the compatibility properties of the δ_v , ω_v , and ϵ_e in the definition of Hurwitz tree, we can glue in \mathbb{Z}/p -covers of the annuli \mathcal{A}_e to form an open unit disc. This disc has a \mathbb{Z}/p -action, and it turns out it even has a faithful G -action that satisfies the requirements of the proposition. \square

More detailed version of proof. Maintain the notations from the sketch above. Let Z_R be a model of \mathbb{P}_K^1 with special fiber \bar{Z} (i.e., a flat R -scheme

whose special fiber is \overline{Z} and whose generic fiber is \mathbb{P}_K^1) such that the specialization of ∞ is $\overline{\infty}'$. Write \mathcal{Z} for the formal completion of Z_R at \overline{Z} . Now, if $v \in V(\Gamma)$ is a vertex other than the root vertex, then the irreducible component \overline{S}_v of \overline{Z} corresponding to v contains r_v points $\overline{s}_1, \dots, \overline{s}_{r_v}$ from among $\overline{\infty}'$ and the singular points of \overline{Z} . These correspond to the edges of Γ incident to v . If $\overline{U}_v \subseteq \overline{S}_v$ is the complement of these points, then the formal subscheme $\mathcal{U}_v \subseteq \mathcal{Z}$ that lifts \overline{U}_v (§A.4) is a projective line with r_v open discs $\mathcal{D}'_{v,1}, \dots, \mathcal{D}'_{v,r_v}$ corresponding to $\overline{w}_1, \dots, \overline{w}_{r_v}$ removed (alternatively, we can view \mathcal{U}_v as a closed disc with $r_v - 1$ open discs removed, see §A.2). Write $\mathcal{U}_v = \mathrm{Spf} A_v$. In [BoW06, §3.4], the data δ_v and ω_v are used to construct a \mathbb{Z}/p -extension B_v/A_v , where B_v is abstractly isomorphic to A_v . Thus, the formal scheme $\mathcal{V}_v := \mathrm{Spf} B_v$ can also be viewed as a formal projective line \mathcal{Y} with r_v open discs $\mathcal{D}_{v,1}, \dots, \mathcal{D}_{v,r_v}$ removed, with each $\mathcal{D}_{v,i}$ lying above the respective $\mathcal{D}'_{v,i}$.

If $C_v \subseteq C$ is the stabilizer of v , then the action of \mathbb{Z}/p on B_v extends to an action of $G_v := \mathbb{Z}/p \rtimes C_v \subseteq G$ on B_v , thus on \mathcal{V}_v . Each $\mathcal{D}_{v,i}$ has a boundary $\partial \mathcal{D}_{v,i}$. If $C_{v,i} \subseteq C_v$ is the stabilizer of the edge of Γ corresponding to $\mathcal{D}'_{v,i}$, then $G_{v,i} := \mathbb{Z}/p \rtimes C_{v,i}$ acts on $\partial \mathcal{D}_{v,i}$. This gives rise to a $G_{v,i}$ -cover $\partial \mathcal{D}_{v,i} \rightarrow \partial \mathcal{D}_{v,i}/G_{v,i}$. It is shown ([BoW06, Lemma 3.3(iii), Prop. 3.9]) that this extension is of type $(\delta_v, h_{v,i} := -\mathrm{ord}_{\overline{w}_i}(\omega_v) - 1, \alpha)$, where $\alpha : C_{v,i} \rightarrow (\mathbb{F}_p)^\times$ is the unique character such that $\alpha^{-h_{v,i}} = \chi|_{C_{v,i}}$ (note that $h_{v,i}$ is called h_i in [BoW06, Prop. 3.9]). There is a sign error in that paper which explains why our $h_{v,i}$ is their h_i).

Now, suppose $e \in E(\Gamma)$ is an edge with stabilizer C_e . Consider the two vertices v_1 and v_2 incident to e . Assume for the moment that neither v_1 nor v_2 is the root vertex. Then we have seen that there is a $G_e := \mathbb{Z}/p \rtimes C_e$ -cover of boundaries of open discs associated to each of v_1 and v_2 , namely, the covers $\partial \mathcal{D}_{v_1,i} \rightarrow \partial \mathcal{D}_{v_1,i}/G_e$, and $\partial \mathcal{D}_{v_2,j} \rightarrow \partial \mathcal{D}_{v_2,j}/G_e$ for the correct i and j corresponding to e . Suppose these covers are of type $(\delta_1, h_1, \alpha_1)$ and $(\delta_2, h_2, \alpha_2)$, respectively. By (vi) of Proposition 7.9, we have $h_2 = -h_1$. This gives us that $\alpha_2 = \alpha_1^{-1}$. By (ii) and (vii) of Proposition 7.9, we have $0 \leq \delta_1, \delta_2 \leq 1$ and

$$\delta_1 - \delta_2 = (p-1)\epsilon_e(-h_1) = (p-1)\epsilon_e h_2.$$

In this situation, [BoW06, §3.5] exhibits an open annulus \mathcal{A}_e with thickness ϵ_e and G_e -action such that one boundary is of type $(\delta_1, h_1, \alpha_1)$ whereas the other is of type $(\delta_1 + (p-1)\epsilon_e h_1, -h_1, \alpha_1^{-1})$. By Lemma 7.6, one can identify the boundaries of \mathcal{A}_e with $\partial \mathcal{D}_{v_1,i}$ and $\partial \mathcal{D}_{v_2,j}$ in

a G_e -equivariant way, and thus we can glue \mathcal{A}_e to \mathcal{V}_{v_1} and \mathcal{V}_{v_2} while respecting the G_e -action.

Alternatively, If e is the edge e_0 , incident to v_1 and the root vertex v_0 , then [BoW06, §3.5] constructs an annulus \mathcal{A}_{e_0} of thickness ϵ_{e_0} with one boundary of type $(\delta_1, h_1, \alpha_1)$ and the other of type $(0, -h_1, \alpha_1^{-1})$. Again, this annulus can be glued along its first boundary to \mathcal{V}_{v_1} while respecting the G_e -action. We know that $-h_1$ is just $\text{ord}_{\infty'}(\omega_{v_1}) + 1$, which by [BoW06, Lemma 3.3(i)] is equal to h .

By gluing the annuli \mathcal{A}_e for all $e \in E(\Gamma)$ to the \mathcal{V}_v for all $v \neq v_0 \in V(\Gamma)$ as above, we obtain an open disc \mathcal{D} . The G -action on \mathcal{D} is given by having G permute the \mathcal{V}_v and \mathcal{A}_e just as it permutes $V(\Gamma)$ and $E(\Gamma)$, and having the stabilizers G_v and G_e of each vertex and edge act as in the construction. One checks that the G -action gives rise to the original Hurwitz tree \mathcal{T} . \square

Corollary 7.12. *If $\mathcal{D} \cong \text{Spec } R[[U]]$ is the open disc with G -action corresponding to the Hurwitz tree in Proposition 7.11, then the reduction of the extension $R[[U]]/R[[U]]^G$ to characteristic p has unique positive lower jump h .*

Proof. By Proposition 7.11, the group G acts on $R[[U]]\{U^{-1}\}$ with conductor h . The reduction of $R[[U]]/R[[U]]^G$ to characteristic p is a G -extension whose associated extension of fraction fields is the reduction of $R[[U]]\{U^{-1}\}/R[[U]]\{U^{-1}\}^G$. By the definition of conductor and the lower numbering, the reduction of $R[[U]]/R[[U]]^G$ has unique lower jump equal to the conductor of $R[[U]]\{U^{-1}\}/R[[U]]\{U^{-1}\}^G$, which is h . \square

7.3.4. Constructing Hurwitz trees. In order to complete the proof of Theorem 7.1, we need only show:

Proposition 7.13. *If $C \cong \mathbb{Z}/m$, if $\chi : C \rightarrow \mathbb{F}_p^\times$ is an injective character, and if $h > 1$ satisfies $h \equiv -1 \pmod{m}$, then there exists a Hurwitz tree of type (C, χ) with conductor h .*

Proof. If $h < p$, we give the construction from [BWZ09, Proof of Prop. 1.4]. We take $\overline{\mathbb{Z}}$ to be *smooth*, so that the graph Γ will only have two vertices v_0 and v_1 , one edge e_0 incident to both, and v_0 is the root vertex. Set $\delta_{v_0} = 0$, $\delta_{v_1} = 1$, and $\epsilon_{e_0} = \frac{1}{h(p-1)}$. Write $h = rm - 1$. Consider $z_1, \dots, z_r \in \mathbb{F}_p^\times$ such that the elements

$$z_{i,j} := \zeta_m^j z_i, \quad 1 \leq i \leq r, \quad 0 \leq j \leq m-1$$

are pairwise distinct (note: $h \neq p-1$). Here ζ_m is an m th root of unity, which we think of as an element of \mathbb{F}_p^\times . Take ∞ and the $z_{i,j}$ to be the

marked points. Then set

$$\omega_{v_1} = \frac{dz}{\prod_{i=1}^r (z^m - z_i^m)}.$$

One shows that this is logarithmic ([BWZ09, Lemma 1.5]). It has a zero at ∞ of order $rm - 2 = h - 1$. Lastly, if c is a generator of \mathbb{Z}/m , take the action of c on $\overline{\mathbb{Z}}$ to be given by $c^*z = \zeta_m z$. It is then an easy exercise to verify properties (i)–(x) of Proposition 7.9.

If $h > p$, then we cannot take $\overline{\mathbb{Z}}$ to be smooth, and the construction is more complicated. See [BWZ09, Proof of Theorem 2.1] and [BoW06, Theorem 4.3]. \square

Remark 7.14. The methods used to prove Theorem 7.1 work equally well to show that all \mathbb{Z}/p -extensions of $k[[t]]$ lift to characteristic zero (this is the subject of Henrio’s thesis [Hen99]). Of course, one would like to extend these methods to lift \mathbb{Z}/p^n - and $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ -extensions of $k[[t]]$ for which the KGB obstruction vanishes. Indeed, there is a notion of Hurwitz tree for actions of $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ on the open unit disc (see [BrW09]). However, there is currently no good analogue of Lemma 7.6 for groups with cyclic p -Sylow subgroups of order greater than p . Without such an analogue, the gluing that is essential for the proof of Proposition 7.11 cannot be performed.

§Appendix A. Non-archimedean geometry

We briefly discuss the constructions and objects in non-archimedean geometry that are used in this paper, taking a rather naïve perspective throughout. For a thorough introduction to formal and rigid geometry, see e.g. [BGR84], or for a quick overview see [Gar98] and [Hen98]. Throughout this appendix, let $R/W(k)$ be finite, and let $K = \text{Frac}(R)$.

A.1. Discs and annuli

We call the scheme $\text{Spec } R[[T]]$ the *open unit disc* (by abuse of language, the term “open unit disc” is used regardless of R). For any algebraically closed field extension K'/K where K' has an absolute value extending that of K , the K' -points of $\text{Spec } R[[T]]$ correspond to the elements $b \in K'$ with $|b| < 1$, by plugging in $T = b$ (these are the values at which all power series in $R[[T]]$ converge). By abuse of language, we will call this set of K' -points the open unit disc as well (we will do the same for the closed disc and annuli described below).

The scheme $\text{Spec } R\{T\}$ is the *closed unit disc* (recall that $R\{T\}$ is the subring of $R[[T]]$ consisting of power series for which the coefficients tend to 0, see §1.4). For K'/K as above, the K' -points of

$\mathrm{Spec} R\{T\}$ correspond to the elements $b \in K'$ with $|b| \leq 1$, by plugging in $T = b$. The *boundary* of the open (or closed) unit disc is the scheme $\mathrm{Spec} R[[T^{-1}]]\{T\}$. Note that this scheme has no K -points. The ring $R[[T^{-1}]]\{T\}$ is a complete DVR with residue field $k((t^{-1}))$.

If $a \in K$ such that $|a| = r$, then $\mathrm{Spec} R[[a^{-1}T]]$ (resp. $\mathrm{Spec} R\{a^{-1}T\}$) is the open (resp. closed) disc of radius r . This is isomorphic to the open (resp. closed) unit disc under the map $T \mapsto a^{-1}T$.

The *open annulus of thickness ϵ* is $\mathrm{Spec} R[[T, U]]/(TU - a)$, where $a \in R$ is such that $v(a) = \epsilon$ (recall that we always set $v(p) = 1$). For K'/K as above, the K' -points of $\mathrm{Spec} R[[T, U]]/(TU - a)$ correspond to those $b \in K'$ with $|a| = p^{-\epsilon} < |b| < 1$ (or, stated in terms of the valuation on K' , that $0 < v(b) < \epsilon$), by plugging in $T = b$. The open annulus has two boundaries, one given by $\mathrm{Spec} R[[T]]\{T^{-1}\}$ and one given by $\mathrm{Spec} R[[U]]\{U^{-1}\}$. Note that two open annuli over R are isomorphic if and only if they have the same thickness.

A.2. Semistable models of curves

Let $X_R \rightarrow \mathrm{Spec} R$ be a semistable curve with smooth generic fiber X and special fiber \overline{X} . If $\overline{x} \in \overline{X}$, then \overline{x} is either a smooth point or a node. If \overline{x} is smooth, then $\mathrm{Spec} \hat{\mathcal{O}}_{X_R, \overline{x}}$ is isomorphic to an open unit disc, whereas if \overline{x} is a node, then $\mathrm{Spec} \hat{\mathcal{O}}_{X_R, \overline{x}}$ is isomorphic to an annulus of some thickness ϵ , which is an intrinsic property of the singularity $\overline{x} \in X_R$ (see, e.g., [Ray99, §2.1.1]).

If X is a projective line, we can give more detail. First assume that X_R is smooth. Then there are (many) elements T in the function field of X such that $K(T) \cong K(X)$ and the local ring at the generic point of the special fiber of X_R is the valuation ring of $K(T)$ corresponding to the Gauss valuation. We say that T is a *coordinate* of X_R (note that there can be many different coordinates for a given model). If T is a coordinate of X_R , if $a, b \in R$, and if $c \in K$, then $T = a$ and $T = b$ coalesce on the special fiber iff $v(a - b) > 0$, and $T = \infty$ and $T = c$ coalesce on the special fiber iff $c \notin R$.

Conversely, if T is any rational function on X such that $K(T) \cong K(X)$, there is a smooth model X_R of X such that T is a coordinate of X_R .

Now, drop the assumption that X_R is smooth. Then \overline{X} is a tree-like configuration of \mathbb{P}_k^1 's. Each irreducible component \overline{W} of the special fiber \overline{X} of X_R corresponds to a smooth model of X , and thus to (many) coordinates T . Such a T is called a *coordinate on \overline{W}* in this case.

Let X_R be a semistable model for $X = \mathbb{P}_K^1$. Let \overline{K} be an algebraic closure of K . Suppose \overline{x} is a smooth point of \overline{X} on the irreducible component \overline{W} . Let T be a coordinate on \overline{W} such that $T = 0$ specializes

to \bar{x} . The complete local ring of \bar{x} in X_R is isomorphic to $R[[T]]$. The set of points of $X(\bar{K})$ that specialize to \bar{x} is the open unit disc $v(T) > 0$.

Now, let \bar{x} be a nodal point of \bar{X} , lying on an irreducible component \bar{W} . Suppose T is a coordinate on \bar{W} such that $T = 0$ specializes to the connected component of $\bar{X} \setminus \{\bar{x}\}$ not containing $\bar{W} \setminus \{\bar{x}\}$. Then the complete local ring of \bar{x} in X_R is isomorphic to $R[[T, U]]/(TU - p^\epsilon)$ where ϵ is the thickness of the annulus corresponding to \bar{x} . The set of points of $X(\bar{K})$ that specialize to $\bar{x} \in \bar{X}$ is the open annulus given by $0 < v(T) < \epsilon$.

Suppose T is a coordinate on a component \bar{W} of \bar{X} , and suppose $T = \infty$ specializes away from \bar{W} . Let \bar{U} be the subset of \bar{W} consisting of smooth points of \bar{X} . Then the set of points of $X(\bar{K})$ that specialize to \bar{U} is a punctured closed unit disc, that is, a closed disc with an open unit disc removed for each point of $(\bar{X} \setminus \{\infty\}) \setminus \bar{U}$.

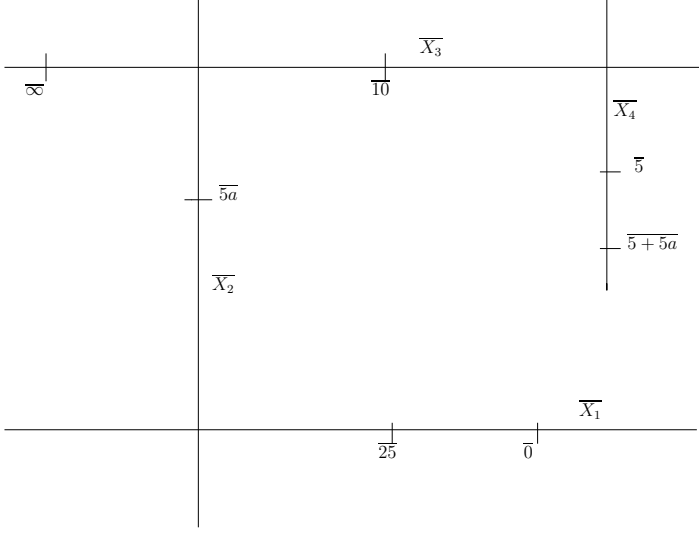
A.3. Stable models and marked discs

Let $X = \mathbb{P}_K^1$, and suppose we are given $x_1, \dots, x_r \in X(K)$, with $r \geq 3$. Then there is a unique semistable model X^{st} for X over R such that the specializations of x_1, \dots, x_r to the special fiber \bar{X} of X^{st} are pairwise distinct, and such that each irreducible component of \bar{X} contains at least three points that are either singular or specializations of an x_i . The R -curve X^{st} is called the *stable model* of the marked curve $(X; x_1, \dots, x_r)$.

Consider the open unit disc $\mathcal{D} := \text{Spec } R[[T]]$, and suppose we are given x_1, \dots, x_r in $\mathcal{D}(K)$, with $r \geq 2$. From §A.1, we can think of x_1, \dots, x_r as elements of the maximal ideal of R . Let X^{st} be the stable model of the marked curve $(\mathbb{P}_K^1; \infty, x_1, \dots, x_r)$, with special fiber \bar{X} (thinking of $\mathbb{P}^1(K)$ as $K \cup \{\infty\}$). We will call X^{st} the *stable model of \mathbb{P}_K^1 corresponding to the marked disc $(\mathcal{D}; x_1, \dots, x_r)$* . Let ∞ be the specialization of ∞ to \bar{X} . Note that the set of points of $\mathbb{P}^1(K)$ that specialize away from ∞ form a closed disc. In particular, each point in this closed disc is an element of the maximal ideal of R (although some elements of the maximal ideal of R might specialize to ∞).

Example A.1. Let $R = W(\overline{\mathbb{F}_5})[a]/(a^2 - 5)$, and consider the marked open unit disc $(\mathcal{D}; 0, 5, 10, 25, 5a, 5 + 5a)$. Then the stable model of $X = \mathbb{P}_K^1$ corresponding to our marked disc has a special fiber \bar{X} with four irreducible components $\bar{X}_1, \bar{X}_2, \bar{X}_3$, and \bar{X}_4 , as in Figure 1 (the specializations of the marked points are marked with overlines in Figure 1).

The following table shows where the \bar{K} -points of X specialize.

Fig. 1. The special fiber \overline{X}

Subscheme \overline{V} of \overline{X}	Points of $X(K)$ that specialize to \overline{V}
$\overline{X}_1 \setminus \overline{X}_2$	$\{x \mid v(x) \geq 2\}$
$\overline{X}_1 \cap \overline{X}_2$	$\{x \mid \frac{3}{2} < v(x) < 2\}$
$\overline{X}_2 \setminus (\overline{X}_1 \cup \overline{X}_3)$	$\{x \mid v(x) = \frac{3}{2}\}$
$\overline{X}_2 \cap \overline{X}_3$	$\{x \mid 1 < v(x) < \frac{3}{2}\}$
$\overline{X}_3 \setminus (\overline{X}_2 \cup \overline{X}_4 \cup \{\infty\})$	$\{x \mid v(x) = 1 \wedge v(x - 5) = 1\}$
$\overline{X}_3 \cap \overline{X}_4$	$\{x \mid 1 < v(x - 5) < \frac{3}{2}\}$
$\overline{X}_4 \setminus \overline{X}_3$	$\{x \mid v(x - 5) \geq \frac{3}{2}\}$
∞	$\{x \mid v(x) < 1\}$

A.4. Formal schemes

Corresponding to each disc or annulus from §A.2, there is a *formal* disc or annulus. In particular, if a disc or annulus is given by $\text{Spec } A$, then the corresponding formal disc or annulus is $\text{Spf } A$. In fact, if \mathcal{X} is the formal completion of X_R at its special fiber (the “formal projective line”), then any subscheme $\overline{V} \subseteq \overline{X}$ lifts to a (canonical) formal subscheme $\mathcal{V} \subseteq \mathcal{X}$, regardless of the genus of X . For instance, take any (Zariski) open subscheme V_R of X_R such that \overline{V} is a closed subscheme of V_R . Then \mathcal{V} is the formal completion of V_R at \overline{V} , viewed as a formal subscheme of \mathcal{X} via the inclusion $V_R \rightarrow X_R$.

§Appendix B. Depth and deformation data

B.1. Depth

Maintain the notation R and K from Appendix A. For any R -scheme X in this appendix, write X_K for its generic fiber and X_k for its special fiber. Likewise, for an R -algebra A , write A_K for $A \otimes_R K$ and A_k for $A \otimes_R k$. We assume throughout that R contains the p th roots of unity.

Given a G -extension $B/R[[T]]$, it is important in §6.6.2 to know how far its reduction is from being separable. In particular, if the reduction is separable, then $B/R[[T]]$ is at least a birational lift of the normalization of the reduction. The most basic interesting case is when $G \cong \mathbb{Z}/p$. We start off by stating a general structure theorem about \mathbb{Z}/p -extensions of certain R -algebras.

Proposition B.1 ([Hen99], Ch. 5, Proposition 1.6). *Let $X = \operatorname{Spec} A$ be a flat affine scheme over R , with relative dimension ≤ 1 and integral fibers. We suppose that A is a factorial R -algebra that is complete with respect to the π -adic valuation (for π a uniformizer of R). Let $Y_K \rightarrow X_K$ be a non-trivial étale \mathbb{Z}/p -cover, given by an equation $y^p = f$, where f is invertible in A_K . Let Y be the normalization of X in $K(Y_K)$; we suppose the special fiber of Y is integral (in particular, reduced). Let η (resp. η') be the generic point of the special fiber of X (resp. Y). The local rings $\mathcal{O}_{X,\eta}$ and $\mathcal{O}_{Y,\eta'}$ are thus discrete valuation rings with uniformizer π . Write δ for the valuation of the different of $\mathcal{O}_{Y,\eta'}/\mathcal{O}_{X,\eta}$. We then have two cases, depending on the value of δ (which always satisfies $0 \leq \delta \leq 1$):*

- *If $\delta = 1$, then $Y \cong \operatorname{Spec} B$, with $B = A[y]/(y^p - u)$, for u a unit in A , not congruent to a p th power modulo π , and unique up to multiplication by a p th power in A^\times . We say that the \mathbb{Z}/p -cover $Y_K \rightarrow X_K$ has multiplicative reduction.*
- *If $0 \leq \delta < 1$, then $\delta = 1 - n(\frac{p-1}{e})$, where n is an integer such that $0 < n \leq e/(p-1)$. Then $Y \cong \operatorname{Spec} B$, with*

$$B = \frac{A[w]}{\left(\frac{(\pi^n w + 1)^p - 1}{\pi^{pn}} - u\right)},$$

for u a unit of A , not congruent to a p th power modulo π . Also, u is unique in the following sense: If an element $u' \in A^\times$ could take the place of u , then there exists $v \in A$ such that

$$\pi^{pn} u' + 1 = (\pi^{pn} u + 1)(\pi^n v + 1)^p.$$

If $\delta > 0$ (resp. $\delta = 0$), we say that the \mathbb{Z}/p -cover $Y_K \rightarrow X_K$ has additive reduction (resp. étale reduction).

Note that the map $Y_k \rightarrow X_k$ above is separable iff $\delta = 0$.

Definition B.2. (i) If δ is as in Proposition B.1, then the number $\frac{p}{p-1}\delta$ is called the *depth* of the extension B/A (or of the map $Y \rightarrow X$).

- (ii) If X is as in Proposition B.1, let $Y_K \rightarrow X_K$ be a \mathbb{Z}/p^n -cover that can be broken up into a tower of \mathbb{Z}/p -covers $Y_K = T_n \rightarrow T_{n-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 = X_K$. Assume that for all i , $1 \leq i \leq n$, the maps $T_{i+1} \rightarrow T_i$ and the normalization A_i of X in T_i satisfy the properties of Proposition B.1 (with A_i playing the role of X and $T_{i+1} \rightarrow T_i$ playing the role of $Y_K \rightarrow X_K$). Then for each T_i/T_{i-1} , we can define δ_i as in Proposition B.1. If the normalization of X in $K(Y_K)$ is $Y = \text{Spec } B$, then the depth of B/A (or of $Y \rightarrow X$) is defined to be

$$\left(\sum_{i=1}^{n-1} \delta_i \right) + \frac{p}{p-1} \delta_n.$$

This is called the *effective different* in [Obu09a] and [Obu10].

B.2. Deformation data

Let $Y \rightarrow X = \text{Spec } A$ be a degree p finite extension as in Proposition B.1, let δ and u be as in Proposition B.1, let \bar{u} be the reduction of u to A_k , and assume $\delta > 0$. Then the \mathbb{Z}/p -cover $Y \rightarrow X$ gives rise to a meromorphic differential form ω on $\text{Spec } A_k$ as follows: if $Y_K \rightarrow X_K$ has multiplicative reduction, set $\omega = d\bar{u}/\bar{u}$. If $Y_K \rightarrow X_K$ has additive reduction, set $\omega = d\bar{u}$. One verifies that ω is independent of the choice of u , and ω is called the *deformation datum* corresponding to the map $Y \rightarrow X$. See [Hen99, Ch. 5] for more details.

Furthermore, suppose that the natural \mathbb{Z}/p -action on Y extends to a $\mathbb{Z}/p \rtimes \mathbb{Z}/m$ action on Y , with $p \nmid m$, which descends to a \mathbb{Z}/m -action on X . Let $\chi : \mathbb{Z}/m \rightarrow \mathbb{F}_p^\times$ be the character given by $c\sigma c^{-1} = \sigma^{\chi(c)}$, for $c \in \mathbb{Z}/m$ and σ any generator of \mathbb{Z}/p . Then, for any $c \in \mathbb{Z}/m$, we have $c^*\omega = \chi(c)\omega$ (this is mentioned in [Wew03, p. 999], and proven in [Obu09b, Construction 3.4]).

Remark B.3. The deformation datum ω is important in constructing Hurwitz trees, see §7.3.1. In particular, it is a characteristic p object that helps retain information that is lost when a branched cover is reduced from characteristic zero. Deformation data are also extremely important in the proof of Theorem 6.28 by Wewers and the author, as they are used to construct the differential equations mentioned in the proof of Proposition 6.22.

Remark B.4. The depth and deformation datum of a map $Y \rightarrow X$ are the two components of Kato's differential Swan conductor, defined in [Kat87] (also see [Bre09, Ch. 1], especially §1.4, for an exposition).

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